CERTAIN TYPES OF INVOLUTORIAL SPACE TRANSFORMATIONS*

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1. **Statement of problem.** Although there are many isolated examples of involutorial space transformations, the only type which has been systematically investigated is the monoidal one.† A (2, 1) correspondence between two spaces (x) and (x') may be expressed by three equations, algebraic and homogeneous in x_1, x_2, x_3, x_4 and all linear and homogeneous in x_1', x_2', x_3', x_4'. With a point P_1 in (x) a unique point P' in (x') is associated, but to P' correspond two points P_1 and P_2. Thus, when either P_1 or P_2 is given the other is uniquely defined. The pairs of points in (x) thus determine an involutorial transformation I in the space (x).

By this method it is possible to discuss various types of involutions and to develop certain properties common to all. Every space involution previously mentioned and having a surface of invariant points is included as a particular case in the present scheme.

2. **Equations of the transformation.** Let the defining equations be

$$\sum a_i x_i' = 0, \tag{1a}$$

$$\sum b_i x_i' = 0, \tag{1b}$$

$$\sum c_i x_i' = 0, \tag{1c}$$

in which a_i, b_i, c_i are polynomials in x_1, x_2, x_3, x_4 of degrees n_1, n_2, n_3 respectively. The surfaces in (x) have points and curves in common which are together equivalent to $n_1 n_2 n_3 - 2$ points.

The image of a plane $\sum \lambda_i x_i' = 0$ in (x') is the surface

$$\sum |a_1 b_2 c_3 \lambda_4| = 0 \tag{2}$$

of order $n_1 + n_2 + n_3 = n$. All the surfaces of the web pass through common basis curves β, and may also have isolated basis points; the basis elements of any three are together equivalent to $n - 2$ common points.

Two surfaces of the web (2) intersect in a variable curve of definite order n'; this curve $c_{n'}$ is the image of a straight line in (x'). The image of a

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† Montesano, *Su le trasformazioni involutorie monoidale*, Istituto Lombardi Rendiconti, series 2, vol. 21 (1888), pp. 579–594.

plane $\sum k_i x_i = 0$ in (x) is a surface $s'_{n'}$ of order n' in (x'). The image in (x') of a straight line in (x) is a curve c'_n of order n.

3. **Surfaces of branch points and of coincidences.** The locus of points in (x') which have two coincident images in (x) is the surface of branch points. It will be designated by $L'(x')$. The image of $L'(x')$ is the surface of coincidences $K(x)$, counted twice. The order of L' is equal to the number of points in which it is met by a straight line c'_1, and also to the number of coincidences on the image curve c_n. If the latter is of genus π, it follows from Zeuthen's formula that the order of $L'(x')$ is $2\pi + 2$.

4. **Fundamental elements.** The image of an isolated basis point in (x) is a curve or a surface. The image of every point of a fundamental curve β is a curve whose order is equal to the multiplicity of β on the surfaces of the web (2). These curves generate a surface B' whose order is equal to the number of points in which $c_{n'}$ meets β. Similarly, there may be fundamental surfaces and curves in (x), images of basis curves and points in (x').

Another kind of basis element in (x') may appear when $n_1 = n_2$, namely those points for which the two equations $\sum a_i x'_i = 0$, $\sum b_i x'_i = 0$ define the same surface in (x).

The jacobian of the web (2) consists of $K(x)$ and of fundamental elements.

5. **The involution I.** The image of a plane s_1 in (x) is a surface s'_n. The image of s'_n, in (x) is s_1 and a residual surface s_N, image of s_1 in the involution I.

The surface s_1 meets $K(x)$ in a plane curve (s_1, K), through which s_N passes. The surface s_N also meets s_1 in a second curve c_d. The curve (s_1, K) is the image of the curve of contact of S'_n, and $L'(x')$.

As in plane geometry* we shall make use of the following lemma:

LEMMA. *The necessary and sufficient condition that the image in (x) of a locus in (x') shall be composite is that the given locus shall touch $L'(x')$ at every common point.*

The curve c_d is the image of the double curve on $s_{n'}$.

The jacobian of the web s_N includes all the fundamental surfaces of I. If B' has B for its image in (x) residual to the basis curve β, then B is a fundamental surface of the involution.

6. **Example; two planes and a quadric surface.** The simplest example of an involution from the present standpoint is furnished by (1) when a_i, b_i are linear in (x), and c_i is quadratic. The image of a plane s'_1 in (x') is a quartic surface s_4 defined by (2). It can be proved† that two surfaces s_4

* Sharpe and Snyder, *Types of* (2, 2) *correspondences between two planes*, these Transactions, vol. 17 (1917), pp. 402–414. See p. 403.

† Salmon, *Modern Higher Algebra*, 4th edition, 1885, Lesson XIX, Art. 271. *Geometry of Three Dimensions*, 4th edition, 1882, p. 321. M. Noether, *Curve multiple di superficie algebriche*, Annali di Matematica, series 2, vol. 5 (1871), pp. 163–166.

intersect in a basis curve β_{11} of order 11 and genus 14, common to all the quartics of the web, and in a variable quintic c_5 of genus 2, meeting β_{11} in 18 points. The curve c_5 is the complete image of a line c_1' of (x').

Let a line c_1' be defined parametrically by

$$\tau x_i' = k_i + l_i \rho$$

in which k, l are constants, and ρ the parameter. By substituting these values for x_i' in (1), and solving for ρ we have equations of the form

$$\frac{\bar{a}}{a} = \frac{\bar{b}}{b} = \frac{\bar{c}}{c} = \rho,$$

in which $a, b, \bar{a}, \bar{b}$ are linear in (x), and $\bar{c}, c$ are both quadratic. The surfaces $(\bar{a}c - a\bar{c}) = 0$, $(\bar{b}c - b\bar{c}) = 0$, $(\bar{b}a - b\bar{a}) = 0$ all contain the image, but the c_9 common to the first two has c_4 not on the third, hence the proper image is a c_5 on a quadric, hence of genus 2.

The image of a point on β_{11} is a straight line, hence the image of the whole curve β_{11} is a ruled surface B_{18}' of order 18.

The surface of coincidences K_{12} is the jacobian of the web of quartics s_4; it is of order 12 and has β_{11} for triple curve. L_6' is of order 6.

The image of a plane s_1 in (x) is a quintic surface s_5'. The complete image of s_4 in (x') consists of s_1' taken twice, and of B', hence we again find that B' is of order 18. The complete image in (x) of s_5' consists of the plane s_1 and of a residual surface s_{19} of order 19, having β_{11} for five-fold curve. The surface s_{19} meets s_1 in the plane curve (s_1, K_{12}), and in a residual curve c_7, image of the double curve of s_5'. The curve c_7 has 11 double points at the points in which β_{11} meets s_1.

A plane section of s_4 is of genus 3, thus (s_1, s_4) is of genus 3; the image of s_4 is s_1', such that to a point of s_4 corresponds a single point of s_1'; similarly, the image of s_1 is s_5', hence the genus of (s_5', s_1') is also 3. Hence the double curve of s_5' is of order 3 and has c_7 for image in (x).

The equation of L_6' is found by expressing the condition that the line $(1a)$, $(1b)$ in (x) shall touch its associated quadric. The form of the equation shows that L_6' has four double points P_1', P_2', P_3', P_4', whose coördinates satisfy the system

$$\begin{Vmatrix} a_1' & a_2' & a_3' & a_4' \\ b_1' & b_2' & b_3' & b_4' \end{Vmatrix} = 0.$$

The double curve of every quintic surface s_5', image of a plane s_1 in (x), passes through all the points P_i'.

The surface L_1' has 35 other double points, the coördinates of which make all the first minors of the determinant in the equation of L_6' vanish. They are the

values of the parameters x_i' for which the line $(1a)$, $(1b)$ is a generator of its associated quadric. These points will be designated by G_i'.

The image s_5' of a plane s_1 touches L_6' along a curve c_{15}' having double points at P_i' and passing simply through G_i'. The image of P_i' is a conic p_i on K_{12}. The image of G_i' is a straight line g_i on K_{12}. Each conic p_i meets β_{11} in 8 points and each line g_i meets β_{11} in 4 points, hence 8 generators of B_{18}' pass through each point P_i' and 4 pass through each point G_i'.

A straight line c_1 meets K_{12} in 12 points; its image c_4' touches L_6' in 12 points. The complete image in (x) of c_4' is c_1 and a residual c_{19} which meets c_1 in 12 points on K_{12}. A line c_1 meeting β_{11} in i points has for image a curve of order $4 - i$, touching L_6' in $3(4 - i)$ points, and i generators of B_{18}'.

The surface B_{18}' has for images in (x) the curve β_{11}, and a surface B_{72} having β_{11} as 19-fold curve, each conic p_i 8-fold and each line g_i 4-fold.

In the involution I each s_4 of the web goes into itself, and every point of K_{12} is invariant. The image of β_{11} is B_{72}, the jacobian of the web s_{19}. The image of any point of p_i is the whole conic p_i and of any point of g_i is the whole line g_i.

Three surfaces of order 19 intersect in 6859 points. For the transformation considered the curve β_{11} is equivalent to 6925 intersections but on account of the four intersections with β_{11} each line g_i counts for -1 intersections, while each conic p_i counts for -8 intersections. Hence the number of variable intersections is $6859 - 6925 + 35 + 32 = 1$. This equation verifies the results found. We have therefore established the existence of the involution I of order 19 having a fundamental five-fold curve β_{11} of order 11 and genus 14, 35 fundamental simple lines which are quadrisecants of β_{11} and 4 fundamental double conics which are octasecants of β_{11}.

7. **Quadrics through a conic.** If $(1b)$, $(1c)$ represent systems of quadrics through the conic $\gamma_2 \equiv x_4 = 0$, $\phi(x_1, x_2, x_3) = 0$ the equations are of the form

$$(1b) \qquad x_4 b + \phi x_1' = 0, \qquad b \equiv \sum b_{ik} x_i x_k',$$

$$(1c) \qquad x_4 c + \phi x_2' = 0, \qquad c \equiv \sum c_{ik} x_i x_k'.$$

The quadric $(1b)$ intersects that defined by $(1c)$ in γ_2 and in a residual conic which is met by the plane $(1a)$ in two variable points. The equations (1) define an involution I distinct from that already considered. Proceeding as before we find that a plane s_1' has for image in (x) a surface s_5 of order 5, having γ_2 for double curve. Also a straight line c_1' goes into a curve c_6 of order 6 and genus 2, meeting γ_2 in 6 points. Two planes s_1' meet in a line c_1'; their image surfaces s_5 meet in γ_2 counted four times, in c_6, and in a basis curve β_{11} of order 11 and genus 11. The curves β_{11}, γ_2 meet in 10 points. These facts

may be expressed by the symbols

$$s_1' \sim s_5 : \gamma_2^2 + \beta_{11}, \qquad [\beta_{11}, \gamma_2] = 10,$$

$$c_1' \sim c_6, \qquad p = 2, \qquad [c_6, \gamma_2] = 6, \qquad [c_6, \beta_{11}] = 16.$$

To find the image of a plane $s_1 \equiv \sum k_i x_i = 0$ we first replace $(1b)$ by

$$(1b') \qquad x_1' c - x_2' b = 0,$$

obtained from $(1b)$ and $(1c)$.

The equation $(1b')$ is quadratic in (x'), instead of linear, as considered in the preceding type. The equations, however, still define a (1, 2) correspondence between the two spaces (x'), (x), as may be seen as follows. Given a point (x), of the two points in (x'), defined by (1), one is always in the fixed plane $x_2' = 0$. If we find the image of s_1 by eliminating x_1, x_2, x_3, x_4 between (1) and $\sum k_i x_i = 0$, x_2' is a factor of the resultant. The other factor, equated to zero defines a surface of order 6, passing simply through the line $\lambda' = x_1' = 0, x_2' = 0$.

Hence $$s_1 \sim s_6' : \lambda'.$$

Similarly, by the method of Art. 6 we find L' to be of order 6, but that it does not contain λ'.

The image of $x_4 = 0$ is a surface of order 6 containing λ' to multiplicity 2. From equations (1) we can see that the image of a point on $x_4 = 0$, not on γ_2 is a point of λ', hence we conclude $\gamma_2 \sim \Gamma_6' : \lambda'^2$. The same result may be obtained by expressing the coördinates of a point of γ_2 in terms of a parameter μ, and eliminating μ between $(1a)$ and $(1b')$, when x_1, x_2, x_3 have been replaced by the quadratic functions of μ. Incidentally it also appears that the image in (x') of a point on γ_2 is a conic in a plane through λ'.

The image of s_5 consists of s_1' taken twice, Γ_6' taken twice, and of B_{16}', the image of β_{11}. The ruled surface B_{16}' contains λ' simply, so that we may write

$$\beta_{11} \sim B_{16}' : \lambda'.$$

To obtain the image of λ', we first consider a point $(0, 0, y_3', y_4')$ on it. From (1), the image is defined by the equations

$$y_3' a_3 + y_4' a_4 = 0, \qquad x_4(y_3' b_3 + y_4' b_4) = 0, \qquad x_4(y_3' c_3 + y_4' c_4) = 0.$$

One image point is the intersection of the three planes

$$y_3' a_3 + y_4' a_4 = 0, \qquad y_3' b_3 + y_4' b_4 = 0, \qquad y_3' c_3 + y_4' b_4 = 0,$$

and the other is the entire line

$$x_4 = 0, \qquad y_3' a_3 + y_4' a_4 = 0.$$

Thus the point (y') is fundamental as to one of its images, and regular as to the other. As the point (y') describes λ', the image point describes a space cubic curve α_3, and the associated line describes a plane pencil in $x_4 = 0$.

Thus we may write

$$\lambda' \sim x_4 = 0 \quad \text{and} \quad \alpha_3 .$$

Moreover, from the preceding results we conclude

$$L_6' \sim K_{15} : \gamma_2^6 + \beta_{11} .$$

The same result is obtained from the jacobian of the web of s_5, images of the planes of (x'). The jacobian consists of K_{15} and of the plane $x_4 = 0$.

The number of points G_i' for which $(1a)$ $(1b')$ define a generator of the quadric $(1c)$ will be denoted by x, and the number of points P_i' for which $(1a)$ $(1b')$ define the same plane in (x) by y.

The image of $s_6' : \lambda'$ in (x) is s_1 and $s_{28} : \gamma_2'' + \beta_{11}^6 + \alpha_3$, image of s_1 in the involution I. Since the image of c_1 in I is c_{28} the curve of intersection of two surfaces s_{28} consists of c_{28} and of fundamental curves. Each line g_i is simple on s_{28} and each conic p_i is double. We obtain

$$x + 8y = 115 .$$

The lines g_i and the conics p_i are simple on K_{15}. Each surface s_{28} meets K_{15} in the plane curve (s_1, K_{15}) and in fundamental curves, hence

$$x + 4y = 75 .$$

Thus $x = 35$, $y = 10$.

The lines g_i meet γ_2 once and β_{11} in three points. The conics p_i meet γ_2 twice, and β_{11} in six points. In the involution I we may now write

$$s_1 \sim s_{28} : \gamma_2^{11} + \beta_{11}^6 + 35g_i + 10p_i^2 + \alpha_3 ,$$

$$\gamma_2 \sim \Gamma_{28} : \gamma_2^{11} + \beta_{11}^6 + 35g_i + 10p_i^2 + \alpha_3^2 ,$$

$$\beta_{11} \sim B_{79} : \gamma_2^{31} + \beta_{11}^{17} + 35g_i^2 + 10p_i^5 + \alpha_3 ,$$

$$\alpha_3 \sim s_1 : \gamma_2 .$$

The jacobian of the involution, of order 108, consists of the plane $x_4 = 0$, of Γ_{28} and of B_{79}.

The result may be stated as follows:

THEOREM: *There exists a space involution of order* 28, *the basis curves consisting of a conic* γ_2 *to multiplicity* 11, *a curve* β_{11} *(of order and genus* 11 *meeting* γ_2 *in* 10 *points) to multiplicity* 6, *thirty-five simple straight lines meeting* γ_2 *and having three points on* β_{11}, *and ten double conics, meeting* γ_2 *twice and* β_{11} *six times.*

8. **Quadrics and cubics.** The next case in order of simplicity is that in

which $(1a)$ is linear, $(1b)$ quadratic and $(1c)$ cubic in (x). If the quadrics have in common a line and a conic meeting it, and the cubics have the line double, and the conic simple, then the residual intersection is necessarily two skew lines belonging to the congruence of lines which meet the line and the conic. If the (x) space is transformed birationally so that the lines of the congruence go into the lines of a bundle, then the involution I is transformed into an involution of the monoidal type already considered by Montesano.

If, however, the basis conic is replaced by two common generators, then the variable intersection is a proper conic, and a new involution results, which we proceed to discuss. Let $\gamma \equiv x_1 = 0,\ x_3 = 0$ and $\delta \equiv x_2 = 0,\ x_4 = 0$ be the two skew lines meeting $\alpha \equiv x_1 = 0,\ x_2 = 0$.

The equations then take the forms

$$(1b) \qquad x_1' x_1 x_2 + x_2' x_1 x_4 + x_3' x_2 x_3 = 0,$$

$$(1c) \quad x_1 x_2 (c_1' x_1 + c_2' x_2) + x_1 x_4 (c_3' x_1 + c_4' x_2) + x_2 x_3 (c_5' x_1 + c_6' x_2) = 0,$$

in which c_i' is linear in (x').

Proceeding as in the previous cases we find

$$s_1' \sim s_6 : \alpha^3 + \gamma^2 + \delta^2 + \beta_{12}, \qquad [\alpha, \beta_{12}] = 6, \qquad [\gamma, \beta_{12}] = [\delta, \beta_{12}] = 5,$$

$$c_1' \sim c_7, \quad p = 2, \quad [c_7, \alpha] = 4, \quad [c_7, \gamma] = [c_7, \delta] = 3, \quad [c_7, \beta_{12}] = 16.$$

There are two fundamental lines in (x'): $\lambda' \equiv x_3' = 0,\ c_6' = 0$, whose image is $x_1 = 0$, and $\mu' \equiv x_2' = 0,\ c_3' = 0$, whose image is $x_2 = 0$. There is also a fundamental point $Q' \equiv (0, 0, 0, 1)$ whose image is a plane cubic curve q_3 in $a_4 = 0$, $[\alpha, q_3] = 2$, $[\beta_{12}, q_3] = 8$, $[\gamma, q_3] = 1$, $[\delta, q_3] = 1$. We also find $L_7' : Q'^2$ and consequently

$$K_{18} : \alpha^9 + \gamma^6 + \delta^6 + \beta_{12}^3 + q_3.$$

Moreover

$$s_1 \sim s_7' : \lambda' + \mu' + Q'^3, \qquad \alpha \sim A_4' : \lambda' + \mu' + Q'^2,$$

$$\gamma \sim \Gamma_3' : \lambda' + Q', \qquad \delta \sim \Delta' : \mu' + Q', \qquad \beta_{12} \sim B_{16}' : \lambda' + \mu' + Q'^8.$$

In the involution I

$$s_1 \sim s_{39} : \alpha^{19} + \gamma^{13} + \delta^{13} + \beta_{12}^7 + q_3^3,$$

$$\alpha \sim A_{22} : \alpha^{11} + \gamma^7 + \delta^8 + \beta_{12}^4 + q_3^2,$$

$$\gamma \sim \Gamma_{17} : \alpha^8 + \gamma^6 + \delta^6 + \beta_{12}^3 + q_3,$$

$$\delta \sim \Delta_{17} : \alpha^8 + \gamma^6 + \delta^6 + \beta_{12}^3 + q_3,$$

$$\beta_{12} \sim B_{94} : \alpha^{46} + \gamma^{31} + \delta^{31} + \beta_{12}^{17} + q_3^8.$$

The jacobian consists of A_{22}, Γ_{17}, Δ_{17}, B_{94} and the two planes $x_1 = 0,\ x_2 = 0$. Further, it is found that $x = 24,\ y = 18$. Ten of the lines g_i are bisecants of

β_{12} and meet γ and δ; the other 14 are bisecants of β_{12} and meet α. The 18 conics p_i meet β_{12} in 5 points, and have one point on α, γ, δ.

9. **Generalization, and a basis of classification.** The next case in order of simplicity is given by the equations

$$H_1 \sum b_{ik} x_i x'_k + H_2 \sum c_{ik} x_i x'_k = 0, \tag{1b}$$

$$H_1 x'_1 + H_2 x'_2 = 0, \tag{1c}$$

in which $H_i = 0$ is a general quadric surface. But more generally we may take for (1b), (1c) any two systems of surfaces which intersect in a variable conic in (x); the same equations defining surfaces which intersect in a variable line in (x'). The equations may be combined to produce two linear equations in (x'), or to produce one linear equation and one quadratic equation in (x). In the cases discussed in Articles 7 and 8, the equations which define the variable conic in (x) explicitly determine in (x') a variable line and an extraneous curve lying on an extraneous surface, which appears as a component of L' and of the image of a plane of (x).

With the exception of such cases, the variable line is defined by two equations of one of the following forms in (x').

I. Both equations linear.
II. One equation linear, the other nonlinear.
III. Neither equation linear.

Of these, I has been completely discussed in Art. 6. In II, the linear equation defines a pencil of planes, and the other a system of surfaces of order n, having the axis of the pencil to multiplicity $n - 1$. Two cases arise, according as the linear equation in (x') is linear or quadratic in (x). In III, the lines belong to a congruence of lines of order one; they consist of the lines meeting a rational curve of order n and its $(n - 1)$-fold secant, or of the bisecants of a space cubic curve.

10. **Surfaces intersecting in two skew lines.** The systems (1b) (1c) may be surfaces intersecting in two variable skew lines. When these lines belong to a rational congruence, a birational point transformation can be found which transforms the congruence into a bundle. The involution is therefore of the monoidal type already considered by Montesano. When the lines do not belong to a congruence, they may be defined by a pair of planes and a system of surfaces of order n, having the line of intersection to multiplicity $n - 1$. The equations have the form

$$x'_1 x_1^2 + x'_2 x_1 x_2 + x'_3 x_2^2 = 0, \tag{1c}$$

$$\sum x'_k b_i = 0, \tag{1b}$$

in which $b_i = 0$ is a surface of order n, having $x_1 = 0$, $x_2 = 0$ to multiplicity $n - 1$.

The surface L' is a quadric cone, hence K is a rational surface. It belongs to a linear system of rational surfaces, of which the images of the planes of (x') is a partial system. Every surface of the system is transformed into itself by the involution I. Within the complete system is a web having a basis point in common. If $s_1 = 0$, $s_2 = 0$, $s_3 = 0$ define three of these surfaces which also pass through the image of the basis point in I, and $s_4 = 0$ is the fourth independent surface of the web, then the transformation

$$y_1 = s_1, \qquad y_2 = s_2, \qquad y_3 = s_3, \qquad y_4 = s_4$$

transforms the involution into one of monoidal type, in which the lines of the bundle with vertex at $(0, 0, 0, 1)$ remain invariant. We now resume the discussion of Types II and III.

11. **Type Π_1. Pencil of quadrics in $(\mathbf{x})$.** The defining equations have the form

$$\sum a_i x_i' = \sum a_i' x_i = 0, \tag{1a}$$

$$\sum b_i' x_i = 0 \quad \text{or} \quad (1b) \quad \sum b_i x_i' = 0, \tag{1b'}$$

$$c_1 x_1' + c_2 x_2' = 0, \tag{1c'}$$

in which $b_i' = 0$ is a surface of order n, having $\lambda' = x_1' = 0$, $x_2' = 0$ for line of multiplicity $n - 1$. The surfaces $b_i = 0$ are of order $2n - 1$, having γ_4, the curve common to the quadrics $c_1 = 0$, $c_2 = 0$ to multiplicity $n - 1$.

The image of a plane s_1', by (2) is found to be s_{2n+2}, a surface of order $2n + 2$, having γ_4 as an n-fold curve, and by the method of Art. 6 the image of a straight line c_1' is a curve c_{2n+3} of order $2n + 3$ and genus $n + 1$. The basis curve β_{6n+1} is of order $6n + 1$ and genus $9n - 3$. The curve c_{2n+3} meets γ_4 in $4n + 4$ points, and β_{6n+1} in $6n + 4$ points. The curves β_{6n+1} and γ_4 meet in $12n - 4$ points. Since the image of c_{2n+3} is c_1' in (x'), it follows that the image B' of β_{6n+1} is of order $6n + 4$, and Γ', the image of γ_4, is of order $4n + 4$.

The image of a plane s_1 is a surface s_{2n+3}' of order $2n + 3$, having the line λ' to multiplicity $2n - 1$. A line c_1 has for image a curve c_{2n+2}'.

From (2) it follows at once that the image of a point $(0, 0, y_3', y_4')$ on λ' is the plane curve

$$a_3 y_3' + a_4 y_4' = 0, \qquad b_3 y_3' + b_4 y_4' = 0$$

of order $2n - 1$ and having four points of multiplicity $n - 1$ on γ_4. The image of λ' is the surface

$$\Lambda_{2n} \equiv a_3 b_4 - a_4 b_3 = 0.$$

It contains β_{6n+1} simply, and γ_4 to multiplicity $n - 1$. The quadric

$c_1 x_1' + c_2 x_2' = 0$ meets β_{6n+1} in 6 points not on γ_4, hence the image plane $c_1 x_1' + c_2 x_2' = 0$ meets B'_{6n+4} in 6 lines, so that λ' is of multiplicity $6n-2$ on B'_{6n+4}.

The equation of Γ', the image of γ_4 may be obtained by eliminating (x) from the equations

$$c_1 = 0, \qquad c_2 = 0, \qquad \sum a_i x_i' = 0, \qquad \sum b_i x_i' = 0.$$

The result is a surface Γ' of order $4(n+1)$, containing λ' to multiplicity $4(n-1)$.

12. **Surfaces of coincidences and of branch points.** From Art. 3 it follows that L'_{2n+4} is of order $2n+4$, and by Art. 6 it has λ' to multiplicity $2n$.

Let x be the number of double points G_i' and y the number of double points P_i'. The surface of coincidences K is the jacobian of the web s_{2n+2}, after removing the component Λ_{2n}. It is of order $6n+4$, contains γ_4 to multiplicity $3n$, and β_{6n+1} to multiplicity 2. It also contains y conics p_i and x lines g_i. A plane section of K_{6n+4} has for image the curve of contact of s'_{2n+3} and L'_{2n+4}. It is of order $8n+6$.

13. **The involution.** The image of s_1 is s'_{2n+3}, having λ' to multiplicity $2n-1$; its image in (x) consists of s_1 and of a surface s_{12n+5} of order $12n+5$ having γ_4 to multiplicity $6n-1$, and β_{6n+1} as a 4-fold curve. A plane s_1 meets its image s_{12n+5} in (s_1, K_{6n+4}) and a residual curve c_{6n+1} of order $6n+1$. This curve has four points of order $3n-1$ on γ_4 and $6n+1$ points of order 2 on β_{6n+1}. It is of genus $9n-5$. Its image is the double curve δ'_{3n} of order $3n$ on s'_{2n+3}. It passes through each point P_i'. The image of y_4 in I is a surface Γ_{24n+8} of order $24n+8$, containing γ_4 to multiplicity $12n-3$, and β_{6n+1} to multiplicity 8. The image of β_{6n+1} is a surface B_{24n+2} of order $24n+8$. It contains γ_4 to multiplicity $12n-2$ and β_{6n+1} to multiplicity 7. The jacobian of I consists of Γ and B.

To determine x and y, it is known that two surfaces s_{12n+5} intersect in a curve c_{12n+5} of order $12n+5$ and in fundamental elements consisting of γ_4, β_{6n+1}, y conics p_i, double on each, and x lines g_i, simple on each. Hence

$$x + 8y = 60n.$$

Similarly, the intersection of s_{12n+5} with K_{5n+4} consists of (s_1, K_{5n+4}) and of fundamental elements

$$x + 4y = 36n + 8.$$

Hence

$$x = 12n + 16, \qquad y = 6n - 2.$$

Each line g_i is a bisecant of γ_4 and of β_{6n+1}. Each conic p_i meets γ_4 and β_{6n+1} in four points.

Three surfaces of order $12n+5$ intersect in $(12n+5)^3$ points. The

equivalence of γ_4 is $1728n^3 + 2160n^2 - 864n + 76$ and of β_{6n+1} is $3456n^2 - 672n + 496$. The decrease in equivalence for the $12n - 4$ intersections of β_{6n+1} and γ_4 is $3456n^2 - 2496n + 448$. Each line g_i decreases the equivalence by 1 and each conic p_i by 8. The total equivalence is therefore $(12n + 5)^3 - 1$ as required.

There exists then an involution of order $12n + 5$ *having a fundamental quartic* γ_4 *of genus* 1 *to multiplicity* $6n - 1$ *and a fundamental curve* β_{6n+1} *of order* $6n + 1$, *genus* $9n - 3$ *to multiplicity* 4, *meeting* γ_4 *in* $12n - 4$ *points. It has also* $12n + 16$ *simple fundamental lines meeting* γ_4, β_{6n+1} *each twice, and* $6n - 2$ *fundamental double conics meeting* γ_4, β_{6n+1} *each in* 4 *points.*

14. **Type Π_2. Pencil of planes in (x).** The equations are

$$(1a) \qquad \sum a_i' x_i = \sum a_i x_i' = 0,$$

$$(1b) \qquad x_1' x_1 + x_2' x_2 = 0,$$

$$(1c) \qquad \sum c_i x_i' = 0, \qquad \text{or} \qquad (1c') \quad \sum c_i H_i = 0,$$

in which c_i is of order $n + 1$ in (x), and of order $n - 1$ in x_1, x_2; c_i' is of order n in (x') and of order $n - 1$ in x_1', x_2'; H_i is quadratic in (x).

Let $\gamma \equiv x_1 = 0, x_2 = 0$ and λ' be defined as before.

We may now write

$s_1' \sim s_{n+3}$: $\quad \gamma^n + \beta_{5n+5}$; $\quad [\beta_{5n+5}, \gamma] = 5n - 2$. $\quad p = 12n + 4$.

$c_1' \sim c_{n+4}$; $\quad$ genus $n + 1$. $\quad [c_{n+4}, \gamma] = n + 2$.

$s_1 \sim s_{n+4}' : \lambda'^{n+1}$.

$c_1 \sim c_{n+3}'$; $\quad [c'_{n+3}, \lambda'] = n + 2$.

$\lambda' \sim \Lambda_{n+2} : \gamma^{n+1} + \beta_{5n+5}$. Point on $\lambda' \sim \lambda_{n+1}$ with $n - 1$ fold point on γ.

$x \sim \Gamma_{n+2}' : \lambda'^{n+1}$. Point on $\gamma \sim \gamma_n'$ with $n - 1$ fold point on λ'.

$\beta_{5n+5} \sim B_{5n+5} : \lambda'^{5n+3}$.

There are $15n + 10$ fundamental lines g_i which meet γ and are trisecants of β_{5n+5} and two fundamental conics p_i which are bisecants of γ and hexasecants of β_{5n+5}. Moreover, we have $K_{3n+6} : \gamma^{3n} + \beta_{5n+5}^2 \sim L_{2n+4}' : \lambda'^{2n}$.

15. **The involution I.** From the preceding results it follows that in the involution

$$s_1 \sim s_{4n+9} : \gamma^{4n+1} + \beta_{5n+5}^3 + (15n + 10) g_i + 2p_i^2.$$

$c_1 \sim c_{4n+9}$; $\quad (4n + 8)$ points on γ and $(12n + 24)$ points on β_{5n+5}.

The fundamental elements are the line γ and the basis curve β_{5n+5}

$$\gamma \sim \Gamma_{4n+8} : \gamma^{4n} + \beta_{5n+6},$$

$$\beta_{5n+5} \sim B_{12n+4} : \gamma^{12n+3} + \beta_{5n+5}^8.$$

A point of β_{5n+5} has for image a cubic curve lying in a plane through γ. Each such plane is invariant, hence the section made on B_{12n+4} by every plane through γ consists of 7 cubic curves. The jacobian of the involution consists of $\Gamma_{4n+2} + B_{12n+24}$.

The equivalence for the intersection of three surfaces, images of planes in the involution, is expressed as follows:

$$(4n+9)^3 = 64n^3 + 432n^2 + 972n + 729.$$

For γ^{4n+1} $-$ $(64n^3 + 432n^2 + 204n + 25)$.
For β^3_{5n+5} $-$ $(\quad 540n^2 + 567n + 729)$.
For $5n-2$ intersections
$+ 540n^2 - 210n$.
For $\sum g_i$ $\quad 15n + 10$.
For $\sum p_i$ $\quad 16$,

making a total of 1, as it should.

For every positive integer n we can now state the following

THEOREM: *There exists an involution of order* $4n+9$, *the basis curves consisting of a line* γ *to multiplicity* $4n+1$, *a curve* β *of order* $5n+5$, *genus* $12n-4$, *meeting* γ *in* $5n-2$ *points, to multiplicity* 3, *of two conics in planes through* γ *and meeting* β *in* 6 *points, to multiplicity* 2, *and finally of* $15n+10$ *simple lines which meet* γ *and are trisecants of* β.

16. **Type III$_1$. Basis curve of odd order. Pencil of planes.** Two cases appear, according as a pencil of planes or of quadrics in (x) is used as one of the defining systems. For the first one, the equations are

$$\sum a_i x'_i = \sum a'_i x_i = 0, \tag{1a}$$

$$x_1 b'_1 + x_2 b'_2 = 0, \tag{1b'}$$

$$b'_1(H_1 x'_1 + H_2 x'_2) + b'_2(H_3 x'_1 + H_4 x'_2) = 0, \tag{1c'}$$

in which $b'_i = 0$ is a surface of order n in (x'), having $\lambda' \equiv x'_1 = 0$, $x'_2 = 0$ for $(n-1)$-fold line, and $H_i = 0$ is a quadric in (x). By means of $(1b')$, $(1c')$ we may write

$$b_1 x'_1 + b_2 x'_2 = 0, \tag{1b}$$

in which $b_1 \equiv H_1 x_2 - H_3 x_1$, $b_2 \equiv H_2 x_2 - H_4 x_1$, and

$$\sum c_i x'_i = 0, \tag{1c}$$

wherein c_i is of order $n-1$ in b_1, b_2, and linear in x_1, x_2. The equation $b_i = 0$ defines a cubic surface containing $\mu \equiv x_1 = 0$, $x_2 = 0$ and γ_2, a curve of order 8, genus 7, meeting μ in 4 points. The surfaces $c_i = 0$ are of order $3n-2$, having μ for n-fold line, and γ_8 to multiplicity $n-1$.

The surfaces $b_1' = 0$, $b_2' = 0$, each of order n, intersect in λ', $(n-1)$-fold on each; and in a rational curve θ'_{2n-1} of order $2n-1$, which meets λ' in $2n-2$ points.

This case offers no new difficulties. The scheme of its characteristic features may be written as follows

$$s_1' \sim s_{3n+2} : \mu^{n+1} + \gamma_8^n + \beta_{7n};\ \gamma_8,\ p = 7\ [\gamma_8, \mu] = 4,$$

$$c_1' \sim c_{3n+3} : p = 2n+1, [c_{3n+3}, \mu] = n+3;\ [c_{3n+3}, \gamma_8] = 8n+4;$$

$$[c_{3n+3}, \beta_7] = 7n+1.$$

$\beta_{7n} \sim B'_{7n+1} : \lambda'^{7n-4} + \theta'^5_{2n-1}$. Five generators concurrent on θ'_{2n-1}.

$s_\lambda \sim s'_{3n+3} : \lambda'^{3n-2} + \theta'^3_{2n-1} p = 0,\ \ [\theta'_{2n-1}, \lambda'] = 2n-2.$

$c_1 \sim c'_{3n+2};\ [c'_{3n+2}, \lambda'] = 3n-1;\ \ [c'_{3n+2},\ \theta'_{2n-1}] = 6n-2.$

$\mu \sim M'_{n+3} : \lambda'^n + \theta'_{2n-1}$. Point on $\mu \sim$ plane μ'_{n+1}.

$\gamma_8 \sim \Gamma'_{8n+4} : \lambda'^{(8n-1)} + \theta'_{2n-1}$. Point on $\gamma_8 \sim$ plane γ'_n.

$\lambda' \sim \Lambda_{3n-1} : \mu^n + \gamma^{n-1} + \beta_{7n}$. Point on $\lambda' \sim$ plane λ_{3n-2}.

$\theta'_{2n-1} \sim \Theta_{6n-2} : \mu^{2n-1} + \gamma_8^{2n-1} + \beta_{7n}$. Point on $\theta'_{2n-1} \sim$ plane θ_3; $[\theta_3, \mu] = 1$.

$K_{3n+7} : \mu^{n+4} + \gamma_8^{n+1} + \beta_{7n} \sim L'_{4n+4} : \lambda'^{4n-2} + \theta'^4_{2n-1}$.

$x = 10n + 20.$

$y = n + 1.$

In the involution I we now have

$$s_1 \sim s_{6n+9} : \mu^{2n+6} + \gamma_8^{2n+1} + \beta_{7n}^2,$$

$$\mu \sim M_{6n+8} : \mu^{2n+5} + \gamma_8^{2n+1} + \beta_{7n}^2,$$

$$\gamma_8 \sim \Gamma_{12n+16} : \mu^{4n+12} + \gamma_8^{4n+1} + \beta_{7n}^4,$$

$$\beta_{7n} \sim B_{6n+8} : \mu^{2n+6} + \gamma_8^{2n+1} + \beta_{7n}.$$

The jacobian of the system consists of M_{6n+8}, Γ_{12n+16}, and B_{6n+8}.

17. **Type III_2. Basis curve of odd order. Pencil of quadrics.** In this case, $(1a)$ is as in the last preceding type, but the others are

$$(1b) \qquad b_1'(x_1 x_1' + x_2 x_2') + b_2'(x_3 x_1' + x_4 x_2') = 0,$$

$$(1c') \qquad H_1 b_1' + H_2 b_2' = 0,$$

in which b_i', H_i have the same meaning as before.

We may write

$$b_1 x_1' + b_2 x_2' = 0,$$

$$b_1 \equiv x_1 H_2 - x_3 H_1, \qquad b_2 \equiv x_2 H_2 - x_4 H_1,$$

and by substituting in (1c'), obtain

$$\sum c_i x_i' = 0, \tag{1c}$$

wherein $c_i = 0$ is a surface in (x), of order $3n-1$, having $\mu_4 \equiv (H_1, H_2)$ to multiplicity n, and γ_5, the residual intersection of $b_1 = 0$, $b_2 = 0$ to multiplicity $n-1$. The quartic μ_4 is of genus 1 and meets γ_5 in 8 points, and γ_5 is of genus 2. The other fundamental lines are defined as before. We now have

$$s_1' \sim s_{3n+3} : \mu_4^{n+1} + \gamma_5^n + \beta_{7n+1}.$$

$$c_1' \sim c_{3n+4} : [c_{3n+4}', \mu_4] = 4n+8, [c_{3n+4}, \gamma_5] = 5n+2,$$

$$[c_{3n+4}, \beta_{7n+1}] = 7n+2.$$

$$s_1 \sim s_{3n+4}' : \lambda'^{3n-1} + \theta_{2n-1}'^3.$$

$$c_1 \sim c_{3n+3}' : [c_{3n+3}', \lambda'] = 3n, [c_{3n+3}', \theta_{2n-1}'] = 6n-2.$$

$\lambda' \sim \Lambda_{3n} : \mu_4^n + \gamma_5^{n-1} + \beta_{7n+1}$. Point on $\lambda' \sim$ plane λ_{3n-1}.

$\mu_4 \sim M_{4n+8}' : \lambda'^{4n} + \theta_{2n-1}'^4$. Point on $\mu_4 \sim$ plane μ_{n+1}' with n-fold point on λ'.

$\theta_{2n-1}' \sim \Theta_{6n-2} : \mu_4^{2n-1} + \gamma_5^{2n-1} + \beta_{7n+1}$. Point on $\theta_{2n-1}' \sim$ plane θ_3.

$\gamma_5 \sim \Gamma_{5n+2}' : \lambda'^{5n-5} + \theta_{2n-1}'$. Point on $\gamma_5 \sim$ plane γ_n'.

$\beta_{7n+1} \sim B_{7n+2}' : \lambda'^{7n-3} + \theta_{2n-1}'^5$. All generators in a plane through λ' concurrent.

$K_{3n+16} : \mu_4^{n+4} + \gamma_5^{n+1} + \beta_{7n+1} \sim L_{4n+4}' : \lambda'^{4n-2} + \theta_{2n-1}'^4$.

$x = 10n+20$. $[g_i, \mu_4] = 2$, $[g_i, \gamma_5] = 1$, $[g_i, \beta_{7n+1}] = 1$.

$y = n+4$. $[p_i, \mu_4] = 4$, $[p_i, \gamma_5] = 2$, $[p_i, \beta_{7n+1}] = 2$.

The features of the involution I are as follows:

$$s_1 \sim s_{6n+17} : \mu_4^{2n+7} + \gamma_5^{2n+2} + \beta_{7n+1}^2,$$

$$\mu_4 \sim M_{12n+32} : \mu_4^{4n+13} + \gamma_5^{4n+4} + \beta_{7n+1}^4,$$

$$\gamma_5 \sim \Gamma_{6n+16} : \mu_4^{2n+7} + \gamma_5^{2n+1} + \beta_{7n+1}^2,$$

$$\beta_{7n+1} \sim B_{6n+16} : \mu_4^{2n+7} + \gamma_5^{2n+2} + \beta_{7n+1}.$$

The jacobian of the involution consists of M_{12n+32}, Γ_{6n+16}, and B_{6n+16}.

18. **Type III$_3$. Basis curve of even order.** Equation (1a) has the same form, while the others are

$$(x_1 x_1' + x_2 x_2') b_2' + x_3 b_3' = 0, \tag{1b}$$

$$(H_1 x_1' + H_2 x_2') b_2' + H_3 b_3' = 0, \tag{1c}$$

in which H_i is quadratic in (x); b_2' is of order $n-1$ in (x') and of order $n-2$ in x_1', x_2'; b_3' is of order n in (x'), and of order $n-1$ in x_1', x_2'. The surfaces $x_1' b_2' = 0$, $x_2' b_2' = 0$, $b_3' = 0$ are all of order n, all have λ' to multiplicity $n-1$ and also pass through a rational curve θ'_{2n-2}. This curve meets λ' in $2n-3$ points.

The surfaces $b_1 \equiv x_2 H_3 - x_3 H_2 = 0$, $b_2 = 0$, $b_3 = 0$ are cubics passing through a curve γ_7 of order 7 and genus 5. By means of (1b) and (1c) we may write

$$b_2 x_1' - b_1 x_2' = 0, \tag{1b'}$$

$$\sum c_i x_i' = 0, \tag{1c'}$$

in which $c_i = 0$ is a surface of order $3n-3$, having γ_7 to multiplicity $n-1$, and the residual conic μ_2 of $b_1 = 0$, $b_2 = 0$.

Proceeding as in the former cases we now find

$s_1' \sim s_{3n+1} : \mu_2^{n-1} + \gamma_7^n + \beta_{7n-3}$.

$c_1' \sim c_{3n+2}$; $[c_{3n+2}, \mu_2] = 2(n-1)$; $[c_{3n+2}, \gamma_7] = 7n+6$;

$[c_{3n+2}, \beta_{7n-3}] = 7n-2$.

$\lambda' \sim \Lambda_{3n-2} : \mu_2^{n-2} + \gamma_7^{n-1} + \beta_{7n-3}$. Point on $\lambda' \sim$ plane λ_{3n-3}.

$s_1 \sim s'_{3n+2} : \lambda'^{3n-3} + \theta'^3_{2(n-1)}$.

$c_1 \sim c'_{3n+1} : [c'_{3n+1}, \lambda'] = 3n-2$.

$\theta'_{2n-2} \sim \Theta_{6n-5} : \mu_2^{2n-2} + \gamma_7^{2n-2} + \beta_{7n-3}$. Point on $\theta'_{2n-2} \sim$ plane θ_3.

$\mu_2 \sim M'_{2(n-1)} : \lambda'^{2(n-2)} + \theta'_{2n-2}$. Point on $\mu_2 \sim$ plane μ'^2_{n-1}.

$\gamma_7 \sim \Gamma'_{7n+6} : \lambda'^{7n-1} + \theta'^7_{2n-2}$. Point on $\gamma_7 \sim \gamma_n'$; $[\gamma_n', \lambda'] = 1$.

$\beta_{7n-3} \sim B'_{7n-2} : \lambda'^{7(n-1)} + \theta'^5_{2n-2}$.

$K_{3n+7} : \mu_2^{n-2} + \gamma_7^{n+2} \sim L'_{4n+4}$.

$x = 10n + 25$.

$y = n + 3$.

The associated involution has the characteristics

$$s_1 \sim s_{6n+10} : \mu_2^{2(n-1)} + \gamma_7^{2n+3} + \beta_{7n-3}^2,$$

$$\gamma_7 \sim \Gamma_{18n+27} : \mu_2^{6(n-1)} + \gamma_7^{6n+8} + \beta_{7n-3}^6,$$

$$\beta_{7n-3} \sim B_{6n+9} : \mu_2^{2(n-1)} + \gamma_7^{2n+3} + \beta_{7n-3}.$$

19. **Basis curve a space cubic curve.** The equations have the form (1a) as

before and the other two are

$$b_1 x_1' + b_2 x_2' + b_3 x_3' = 0, \tag{1b}$$

$$b_1 x_2' + b_2 x_3' + b_3 x_4' = 0, \tag{1c}$$

in which b_i is defined in Art. 18. The lines in (x') defined by (1b), (1c) are bisecants of the cubic curve common to the quadrics

$$x_1' x_3' - x_2'^2 = 0, \qquad x_1' x_4' - x_2' x_3' = 0, \qquad x_2' x_4' - x_3'^2 = 0.$$

It will be denoted by θ_3'. The characteristics of the (1, 2) transformation are

$s_1' \sim s_7 : \gamma_7^2 + \beta_{13}$; $[\beta_{13}, \gamma_7] = 32$. β_{13} is of genus 11.

$c_1' \sim c_8$; $[c_8, \gamma_7] = 20$; $[c_8, \beta_{13}] = 14$. $p = 4$.

$s_1 \sim s_8' : \theta_3'^3$.

$c_1 \sim c_7'$; $[c_7', \theta_3'] = 11$.

$\gamma_7 \sim \Gamma_{20}' : \theta_3'^7$. Point on $\gamma_7 \sim \gamma_2'$.

$\beta_{13} \sim B_{14}' : \theta_3'^7$.

$\theta_3' \sim \Theta_{11} : \gamma_7^3 + \beta_{13}^2$. Point on $\theta_3' \sim \theta_3$.

$K_{15} : \gamma_7^4 + \beta_{13} \sim L_{10}' : \theta_3'^4$.

$x = 35$.

$y = 4$.

The characteristics of the involution are

$$s_1 \sim s_{22} : \gamma_7^7 + \beta_{13}^2,$$

$$\gamma_7 \sim \Gamma_{63} : \gamma_7^{20} + \beta_{13}^6,$$

$$\beta_{13} \sim B_{21} : \gamma_7^7 + \beta_{13}.$$

We may therefore state the following

THEOREM: *There exists an involution of order* 22, *having for fundamental curves a curve* γ *of order* 7 *and genus* 5 *seven fold, a curve* β *of order* 13 *and genus* 11, *two fold. The fundamental curves* γ *and* β *meet in* 32 *points. In addition there are* 35 *simple basis lines and* 4 *double conics. The lines meet* γ *in* 3 *points, and* β *in one. The conics meet* γ *in* 6 *points and* β *in two.*

21. **Reduction of a congruence to a bundle.** In the various cases defined by lines belonging to a rational congruence in (x') we now make the following transformation:

Let

$$y_1' = x_1' b_1', \qquad y_2' = x_2' b_1', \qquad y_3' = x_1' b_2'$$

if the basis curve is of odd order;

$$y_1' + x_1' b_2', \qquad y_2' = x_2' b_2', \qquad y_3' = b_3'$$

if the basis curve is of even order, and

$$y_1' = x_1' x_4' - x_2' x_3', \qquad y_2' = x_1' x_2' - x_2'^2, \qquad y_3' = x_2' x_4' - x_3'^2$$

if the basis curve is a space cubic.

The transformation

$$z_1' = y_1' x_4', \qquad z_2' = y_2' x_4', \qquad z_3' = y_3' x_4', \qquad z_4' = \sum a_{ik} x_i' y_k'$$

is birational and transforms the congruence into a bundle.

The equations of III_1 become

$$x_1 z_1' + x_2 z_3' = 0,$$

$$H_1 z_1'^2 + H_2 z_1' z_2' + H_3 z_1' z_3' + H_4 z_2' z_3' = 0,$$

$$\sum x_i M_k' = 0,$$

in which $M_k' = 0$ is a monoid with vertex at $(0, 0, 0, 1)$.

In the equations of III_2 the coefficients x_i, H_i in the first two should be interchanged, the third remaining as before. In the two remaining types, the first two equations are replaced by

$$x_1 z_1' + x_2 z_2' + x_3 z_3' = 0,$$

$$H_1 z_1' + H_2 z_2' + H_3 z_3' = 0,$$

and the third has the same form.

22. **Three general types of (1, 2) correspondences.** Let f_1', f_2', f_3' be three independent cremona functions of y_1', y_2', y_3'. The equation

$$\sum a_{ik} x_i x_k' = 0 \tag{1a}$$

and either

$$x_1 f_1' + x_2 f_2' = 0, \qquad \sum H_i F_i' = 0,$$

or

$$\sum x_i F_i' = 0, \qquad H_1 f_1' + H_2 f_2' = 0,$$

or

$$x_1 f_1' + x_2 f_2' + x_3 f_3' = 0, \qquad H_1 f_1' + H_2 f_2' + H_3 f_3' = 0,$$

in which F_i' is of order k in f_1', f_2', f_3' and linear in f_3' are general types of which the various forms of Type III are special cases. The transformation

$$z_1' = f_1' x_4', \qquad z_2' = x_4' f_2', \qquad z_3' = x_4' f_3', \qquad z_4' = \sum a_{ik} x_i' f_k'$$

reduces them to forms similar to those of the preceding articles. All such correspondences are therefore included as special cases of the type in which a point in (x') is determined by the intersection of a line of a bundle with a monoid.

This discussion completes the classification of those involutions in which one of the equations (1) defining the (1, 2) correspondence between (x) and (x') is bilinear.

23. **Forms containing general quadrics.** When one equation (1) represents a general quadric in (x), the other two must define a general line. As in Art. 9 these equations may be both linear, one linear, or neither linear. The first two cases have already been treated. In the third case the lines belong to a congruence of order one, which can be mapped upon a bundle, thus reducing the involution to the type already discussed by Montesano.

24. **Forms having basis points or curves.** Another paper is completed in which the quadric surfaces have basis curves or points in common; certain other types will be treated, in which all of the defining equations are of degree higher than the second, thus necessarily having basis elements.

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ON A NEW TREATMENT OF THEOREMS OF FINITENESS*

BY

OLIVER EDMUNDS GLENN

Propositions relative to the finiteness of algebraic invariant systems can be treated under the theory of invariants of the type which involves the parameters of the transformations T to which a binary form

$$f_m = (a_0, \cdots, a_m)(x_1, x_2)^m$$

is subjected. The algorithms of this theory are of extensive content. The full discussion of these functions comprehends not only the ordinary binary algebraic invariant theory as one of its phases, but, as was made clear in detail by the present author, in a former paper,† it comprehends, also, the general theory of orthogonal concomitants, of booleans, and of still other invariant systems appertaining to important special sets‡ of binary substitutions. It might have been anticipated, I presume, that an invariant theory based upon considerations connected with the poles of the transformations, as the present theory is based, would be likely to be deeply grounded, but, prior to my own publication on this subject, no one seems to have adopted this point of view.

This paper contains a new and very simple proof of Gordan's theorem, and a determination of the complete system of concomitants in the domain $R(1, T, 0)$, of the binary quintic, carrying with it as one special case, for instance, the previously undetermined complete system of orthogonal concomitants of the quintic form.

As far as my knowledge goes, general proofs of the finiteness of orthogonal or of boolean systems have been given by no one, although these were among the first invariants discovered.§ Apparently this was because researchers long took the erroneous view that these theories were fully included in that of simultaneous concomitants of the ordinary formal type. In this paper I prove these theorems for the general order m, and show that the systems are given by the finite set of irreducible solutions of a definite linear diophantine equation.

* Presented to the Society, December, 1917, and September, 1918.

† These Transactions, vol. 18 (1917), p. 443.

‡ Annals of Mathematics, vol. 20 (1918), p. 125.

§ Boole, Cambridge Mathematical Journal, vol. 3 (1843), p. 1.

1. Certain fundamental systems of the quintic

The poles of the general transformation

$$T: \begin{cases} x_1 = \alpha_1 x_1' + \alpha_2 x_2', \\ x_2 = \beta_0 x_1' + \beta_1 x_2', \end{cases}$$

are the roots of the linear forms

$$(1_1) \qquad f_{\pm 1} = 2\beta_0 x_1 + (\beta_1 - \alpha_1 \pm \Delta) x_2,$$

the latter being, accordingly, covariants of T. When f_m is expanded in terms of $f_{\pm 1}$ as arguments:

$$(1) \qquad f_m = \sum_{i=0}^{m} \binom{m}{i} \phi_{m-2i} f_{+1}^{m-i} f_{-1}^{i},$$

the coefficients ϕ of the expansion are linear invariants of f_m under T in the domain $R(1, T, \Delta)$, the notation for this domain indicating polynomials rational in the coefficients of T, in

$$\Delta = \sqrt{(\alpha_1 - \beta_1)^2 + 4\alpha_2 \beta_0},$$

and rational numerically. The invariant relations are

$$(2) \qquad f_{+1}' = \rho_1^{-1} f_{+1}, \qquad f_{-1}' = \rho_{-1}^{-1} f_{-1},$$

where

$$\rho_{\pm 1} = \tfrac{1}{2}(\alpha_1 + \beta_1 \pm \Delta), \qquad \rho_{+1}\rho_{-1} = D = \alpha_1\beta_1 - \alpha_2\beta_0;$$

and

$$(3) \qquad \phi_{m-2i}' = \rho_1^{m-2i} D^i \phi_{m-2i} \qquad (i = 0, \cdots, m).$$

The invariants ϕ_{m-2i} can be given an explicit representation if a symbolical notation is adopted. Thus

$$f = a_x^m = (a_1 x_1 + a_2 x_2)^m = b_x^m.$$

and the inverse of the transformations

$$f_{\pm 1} = 2\beta_0 x_1 + (\beta_1 - \alpha_1 \pm \Delta) x_2,$$

is

$$T: \begin{cases} x_1 = (-4\beta_0\Delta)^{-1}[(\gamma_1 - \Delta) f_{+1} - (\gamma_1 + \Delta) f_{-1}], \\ x_2 = (-4\beta_0\Delta)^{-1}[2\beta_0(-f_{+1} + f_{-1})] \end{cases} \qquad (\gamma_1 = \beta_1 - \alpha_1).$$

Hence the expansion (1) is given by

$$f = (-4\beta_0\Delta)^{-m}\{[(\gamma_1 - \Delta) a_1 - 2\beta_0 a_2] f_{+1} + [-(\gamma_1 + \Delta) a_1 + 2\beta_0 a_2] f_{-1}\}^m,$$

that is

$$(4) \qquad \phi_{m-2i} = (-1)^i [(\gamma_1 - \Delta) a_1 - 2\beta_0 a_2]^{m-i} [(\gamma_1 + \Delta) a_1 - 2\beta_0 a_2]^i (-4\beta_0\Delta)^{-m}, \qquad (i = 0, \cdots, m).$$

This notation will be especially useful in the second section of this paper.

If we differentiate ϕ_{m-2i} according to the operator

$$a_1\frac{\partial}{\partial a_2}+2\beta_0\frac{\partial}{\partial\gamma_1}-\gamma_1\frac{\partial}{\partial\alpha_2},$$

the result is found to vanish. Hence, since

$$a_1\frac{\partial}{\partial a_2}=a_1\left[\frac{\partial}{\partial a_0}\frac{\partial a_0}{\partial a_2}+\frac{\partial}{\partial a_1}\frac{\partial a_1}{\partial a_2}+\cdots+\frac{\partial}{\partial a_m}\frac{\partial a_m}{\partial a_2}\right],$$

the invariant ϕ_{m-2i} satisfies the differential equation

$$\left(a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+\cdots+ma_{m-1}\frac{\partial}{\partial a_m}+2\beta_0\frac{\partial}{\partial\gamma_1}-\gamma_1\frac{\partial}{\partial\alpha_2}\right)\phi_{m-2i}=0.$$

Every invariantive function, rational or irrational, of the coefficients and variables of f_m, of the coefficients of T, and of Δ, is expressible in terms of the ϕ-s and of $f_{\pm1}$. We have only to employ the inverse of the set of linear equations (in $a_0, \cdots, a_m$)

$$\phi_{m-2i}=\phi_{m-2i}(a_0, \cdots, a_m) \qquad (i=0, \cdots, m).$$

In the domain $R(1, T, \Delta)$ a complete system is composed of ϕ_{m-2i} $(i=0, \cdots, m), f_{+1}, f_{-1}$.

In the domain $R(1, T, 0)$, rational in the coefficients of T, a complete system is in one-to-one correspondence with those invariantive products

$$P=\prod_{i=0}^{m}\phi_{m-2i}^{x_i}f_{+1}^{\sigma_1}f_{-1}^{\sigma_2} \tag{5}$$

for which the exponent of ρ_1 in the invariant relation

$$P'=\rho_1^a D^b P$$

is zero, and for this reason complete systems in this domain are given by the complete set of irreducible solutions in positive integers of the linear diophantine equation

$$a=\prod_{i=0}^{m}(m-2i)x_i-\sigma_1+\sigma_2=0. \tag{6}$$

The particularization of T which changes the systems belonging to the domain $R(1, T, 0)$ into boolean concomitants is given by

$$\alpha_1=\frac{\sin(\omega-\alpha)}{\sin\omega}, \qquad \alpha_2=\frac{\sin(\omega-\beta)}{\sin\omega}, \qquad \beta_0=\frac{\sin\alpha}{\sin\omega}, \qquad \beta_1=\frac{\sin\beta}{\sin\omega}.$$

When $\omega=\frac{1}{2}\pi$ the latter systems become orthogonal. We have now proved the following

THEOREM. *Complete systems in the domain* $R(1, T, 0)$, *booleans, ortho-*

gonal systems, as well as concomitants of f_m under the so-called relativity transformations*

$$T: \begin{cases} t = \dfrac{1}{\sqrt{1-\beta^2}}\left(t' + \dfrac{v}{c^2}x'\right), \\ x = \dfrac{1}{\sqrt{1-\beta^2}}(x' + vt'), \end{cases}$$

are finite, and are given by the finite set of irreducible solutions in positive integers of the equation (6).

In the instance $m = 5$, therefore, we have to solve the equation

$$5x_0 + 3x_1 + x_2 + \sigma_2 = \sigma_1 + x_3 + 3x_4 + 5x_5 \tag{7}$$

and if an irreducible solution is given by the ξ-s in the first row in the table following, another irreducible solution representing the product conjugate to the first product is furnished by the second row:

x_0	x_1	x_2	σ_2	x_3	x_4	x_5	σ_1
ξ_0	ξ_1	ξ_2	s_2	ξ_3	ξ_4	ξ_5	s_1
ξ_5	ξ_4	ξ_3	s_1	ξ_2	ξ_1	ξ_0	s_2

If the two products P, Q thus found are combined, giving $P \pm Q = \alpha_{\pm 1}$, then the $\alpha_{\pm 1}$, deprived of certain irrelevant constant factors, are members of the fundamental system. I have carried out the work of solving (7) and I give in the table below the set of irreducible solutions found. For convenience in reading off the concomitants $P \pm Q$ a solution and its conjugate are juxtaposed under a double notation, as $l_{\pm 1}$, the concomitants $l_{\pm 1}$ being, evidently,

$$l_{\pm 1} = \phi_3 \phi_{-1}^2 f_{+1} \pm \phi_{-3} \phi_1^2 f_{-1}. \tag{8}$$

The number of invariants and covariants in this fundamental system including $f_{+1}f_{-1}$, is 42, of which 15 are pure invariants. The solutions furnishing the latter are placed in the top rows of the table.

A few remarks should perhaps be added as to the method of solving (7). Transposition of terms reduces the equation to

$$5x + 3y + z = 0, \tag{9}$$

where

$$x_0 - x_5 = x, \qquad x_1 - x_4 = y, \qquad x_2 - x_3 - \sigma_1 + \sigma_2 = z. \tag{9_1}$$

The equation (9) is now solved for its irreducible solutions in both positive and negative integers x, y, z, and to each of these solutions correspond sets for (7), given by (9_1). The problem is thus divided into a large number of simple mutually exclusive sub-problems.

* R. D. Carmichael, *Theory of Relativity*, p. 44.

	x_0	x_1	x_2	x_3	x_4	x_5	σ_1	σ_2
a.....	1					1		
b.....		1			1			
c.....			1	1				
$d_{\pm 1}$...		1	3	3	1			
$e_{\pm 1}$....	1		5	5		1		
$g_{\pm 1}$...	1	1	2	2	1	1		
$h_{\pm 1}$...	1	2	1	1	2	1		
$i_{\pm 1}$....	3	5			5	3		
$j_{\pm 1}$....	2	3	1	1	3	2		
$k_{\pm 1}$...			1	1			1	1
$l_{\pm 1}$....		1	2	2	1		1	1
$m_{\pm 1}$...		1	1	1	1		2	2
$n_{\pm 1}$...		1			1		3	3
$o_{\pm 1}$...	1		4	4		1	1	1
$p_{\pm 1}$...	1		3	3		1	2	2
$q_{\pm 1}$...	1		2	2		1	3	3
$r_{\pm 1}$ ...	1		1	1		1	4	4
$s_{\pm 1}$ ...	1					1	5	5
$t_{\pm 1}$ ...	1	1	1	1	1	1	1	1
$u_{\pm 1}$...	1	1			1	1	2	2
$v_{\pm 1}$ ...	1	2			2	1	1	1
$w_{\pm 1}$...	2	3			3	2	1	1

2. The finiteness of algebraic systems

Consider now the following well-known lemma on systems of polynomials, first proved by Hilbert: If an infinite system of homogeneous forms in n variables satisfies any law sufficiently explicit to locate an arbitrarily chosen homogeneous form within or without the system, then there always exists a set of these forms, finite in number, as $F_1, \cdots, F_s$, each within the system,

and such that any form F belonging to the system is expressible in the form

$$F = T_1 F_1 + \cdots + T_s F_s, \tag{10}$$

where $T_1, \cdots, T_s$ are also homogeneous polynomials in the same n variables but do not necessarily belong to the original system.*

The generality of this theorem with regard to the law or laws which determine the system of forms under consideration makes it applicable to a variety of problems, but the possibilities in connection with the study of particular given laws of formation and the systems appertaining have not been much exploited in literature.

When we apply the methods of this paper to systems of concomitants in the domain $R(1, 0, 0)$, functions which are free from the coefficients of the transformation T and therefore are the ordinary rational integral concomitants of f_m, we discover a very interesting application of this lemma. The character of (1) as a typical representation of f_m immediately shows that any concomitant C of f_m in $R(1, 0, 0)$ is a homogeneous polynomial F in the invariants ϕ_{m-2i} $(i = 0, \cdots, m)$, f_{+1}, f_{-1}, and formed according to a law sufficiently definite to define a system of polynomials in the technical sense of the lemma. This law is that F shall be such a polynomial that, although its variables ϕ_{m-2i}, f_{+1}, f_{-1} belong to the domain $R(1, T, \Delta)$, it simplifies by multiplying out into an expression (in $x_1, x_2, a_0, \cdots, a_m$) belonging to the domain $R(1, 0, 0)$, the parameters α_1, α_2, β_0, β_1 disappearing by cancellation from numerator and denominator.

Illustration. If $m = 2$ we have; $(\gamma_1 = \beta_1 - \alpha_1)$,

$$\phi_{\pm 2} = (2^3 \beta_0^2 \Delta^2)^{-1} [(\gamma_1^2 + 2\alpha_2 \beta_0 \mp \gamma_1 \Delta) a_0 + 2\beta_0(-\gamma_1 \pm \Delta) a_1 + 2\beta_0^2 a_2],$$

$$\phi_0 = (2\beta_0 \Delta^2)^{-1} [\alpha_2 a_0 + \gamma_1 a_1 - \beta_0 a_2],$$

and

$$4\beta_0^2 \Delta^2 (\phi_0^2 - 4\phi_2 \phi_{-2}) \equiv a_1^2 - a_0 a_2 \quad (= F).$$

We can now prove that the system of concomitants belonging to the domain $R(1, 0, 0)$ is finite (Gordan's theorem). For, suppose that F is any such concomitant. It is a homogeneous polynomial in ϕ_{m-2i} $(i = 0, \cdots, m)$, f_{+1}, f_{-1}, as above. Hence there exists a finite set of polynomials in ϕ_{m-2i}, f_{+1}, f_{-1}, as $F_1, \cdots, F_s$, each simplifying by multiplication into a concomitant belonging to the domain $R(1, 0, 0)$, such that

$$F = T_1 F_1 + \cdots + T_s F_s, \tag{10}$$

and the polynomials T_i, being homogeneous in the invariants ϕ_{m-2i}, $f_{\pm 1}$, are concomitants which may or may not belong to the domain $R(1, 0, 0)$.

*Mathematische Annalen, vol. 36 (1890), p. 473.

We can now prove that the T-s do, however, belong, in fact, to this domain. This will be equivalent to a proof of the finiteness theorem as each T function can then be expressed linearly in terms of $F_1, \cdots, F_s$ and the process repeated until F is reduced to a polynomial in the forms of the finite fundamental system $F_1, \cdots, F_s$.

The following propositions should now be noted: (*a*) The selection of the set of concomitants F_i in (10) is not unique. The expression of F as a polynomial in $f_{+1}, f_{-1}, \phi_{m-2i}$ $(i = 0, \cdots, m)$ is, on the other hand, unique, since we can obtain this expression (eq. (10)) by the substitution for $x_1, x_2, a_0, \cdots, a_m$ in F, the linear functions τ in f_1, f_{-1}, and the linear functions in $\phi_m, \cdots, \phi_{-m}$ obtained by solving for the inverse of equations (4). Note that the equations (4) are consistent for this solution. If the determinant of this linear system were zero, the ϕ_{m-2i} would be linearly connected; then $a_0, \cdots, a_m$ would be linearly dependent. But these are arbitrary.

Although F is a unique polynomial P in $f_1, f_{-1}, \phi_m, \cdots, \phi_{-m}$ and (10) is merely a grouping of the terms of P into components such that $F_1, \cdots, F_s$ are concomitants which simplify into the domain $R(1, 0, 0)$, nevertheless, even after the non-unique set $F_1, \cdots, F_s$ is definitely selected, the set of auxiliary factors $T_1, \cdots, T_s$ is not necessarily unique.*

(*b*) The linear transformation T is generated by combination of the following particular transformations:

$$T_n : x_1 = \xi + n\eta, \qquad x_2 = \eta;$$

$$S_k : x_1 = \xi, \qquad x_2 = k\eta \ (k \neq 0);$$

$$V : x_1 = -\eta, \qquad x_2 = \xi.$$

Invariancy under V is equivalent to the property of symmetry of invariants under the permutations $(a_0 a_m)(a_1 a_{m-1}) \cdots (x_1 x_2)$. The transformation S_k represents isobarism, and T_n, seminvariance. It follows that a homogeneous polynomial in $a_0, \cdots, a_m, x_1, x_2$ which possesses the properties of symmetry, isobarism, and seminvariance is a concomitant under T, and conversely.

(*c*) The equations of τ, looked upon as linear transformations are as general as those of T.

(*d*) If f is transformed into $f' = (a_0', \cdots, a_m' \mid x_1, x_2)^m$ by a general linear transformation t and if $I' = I(a_0', \cdots, a_m'; x_1', x_2')$ is any homogeneous polynomial in its arguments which possesses the technical properties of symmetry, isobarism, and seminvariance, then I' is the left-hand member of the invariant relation of an invariant I, viz.,

$$I(a_0', \cdots; x_1', x_2') = D^\lambda I(a_0, \cdots; x_1, x_2). \tag{11}$$

* Cf. some theory by the present writer, on expressing a polynomial in terms of others, in these Transactions, vol. 15 (1914), p. 77, and vol. 18 (1917), p. 443.

To prove this, notice that the assumed properties of symmetry, isobarism, and seminvariance are sufficient conditions for the invariance of I' under the transformation of f' by a general linear transformation t'. But if f' has a formal invariant I', under t', then, by correspondence, f has a formal invariant $I(a_0, \cdots, a_m; x_1, x_2)$ under t. Hence I' cannot be other than the transformed function of $I(a_0, \cdots, a_m; x_1, x_2)$ under transformation of f by t, which was to be proved.

(*e*) The concomitants T_i belong to the domain $R(1, T, 0)$ if not to $R(1, 0, 0)$. For, if f' is the transformed of f by T, the relation for f' analogous to (10) is of the type

$$(\alpha\beta)^k \cdot F = T_1'(\alpha\beta)^{k_1} F_1 + \cdots + T_s'(\alpha\beta)^{k_s} F_s \quad [(\alpha\beta) = D],$$

and this relation is necessarily identical with (10). Thus the invariant relation for T_i is of the form

$$T_i' = (\alpha\beta)^{k_i'} T_i,$$

where k_i' is a positive integer. Since the multiplier in this invariant relation involves D, and neither of its irrational factors ρ_1, ρ_{-1}, this amounts to a sufficient condition that T_i belong to $R(1, T, 0)$.

Consider next the equation (1) of Section 1. This form $f' = f_m$ is the result of transforming f_m by a binary substitution τ of general type (cf. proposition (*c*)). We wish to show that the auxiliary factors $T_1, \cdots, T_s$ in (10) are functions I' of proposition (*d*), i. e., they are functions of ϕ_{m-2i} $(i = 0, \cdots, m)$, f_{+1}, f_{-1} which possess the formal properties of symmetry, isobarism, and seminvariance.

(*A*) *Symmetry.* That T_i is unaltered, save possibly by sign, by the permutational substitution $S = (\phi_m \phi_{-m})(\phi_{m-2} \phi_{-m+2}) \cdots (f_1 f_{-1})$ follows immediately from the fact that T_i belongs to the domain $R(1, T, 0)$. For, it must be a linear function of combinations $P + Q$, or of $P - Q$, where Q is a product of powers of ϕ_{m-2i} $(i = 0, \cdots, m)$, f_{+1}, f_{-1} and P is the product conjugate to Q. But the substitution S interchanges P and Q (cf. (8)).

(*B*) *Isobarism.* The isobaric property of T_i implies, in this instance, its invariance under the transformations

$$\phi'_{m-2i} = k^i \phi_{m-2i}; \qquad f'_{+1} = f_{+1}, \qquad f'_{-1} = k^{-1} f_{-1} \tag{12}$$

(cf. proposition (*b*)).

Now a representative term of T_i is (cf. (5))

$$P = \phi_m^{x_0} \phi_{m-2}^{x_1} \phi_{m-4}^{x_2} \cdots f_{+1}^{\sigma_1} f_{-1}^{\sigma_2},$$

and both $\sigma_1 + \sigma_2$ and $x_0 + x_1 + \cdots + x_m$ are invariable. If the transformations (12) are applied to P the exponent of the power of k which is a factor of the result is

$$w = x_1 + 2x_2 + \cdots + mx_m - \sigma_2.$$

But from equation (6),

$$w = \tfrac{1}{2}\,[\,m\,(x_0 + x_1 + \cdots + x_m) - (\sigma_1 + \sigma_2)\,],$$

and this number is evidently invariable for the various terms P of T_i. Hence the isobaric property is proved.

(*C*) *Seminvariance.* The seminvariance of T_i is the property of remaining absolutely unaltered when we apply to it those substitutions upon ϕ_{m-2i} $(i = 0, \cdots, m)$, f_{+1}, f_{-1} which are induced by the transformations T_n of proposition (*b*), viz.,

$$f_{+1} = f'_{+1} + nf'_{-1}, \qquad f_{-1} = f'_{-1}, \tag{13}$$

applied to f_m of (1).

Let us suppose now that n is infinitesimal. Then, by a well-known bit of elementary analysis which need not be repeated here, the result of transforming T_i by the induced transformations is

$$T'_i = T_i + n\left[\phi_m \frac{\partial}{\partial \phi_{m-2}} + 2\phi_{m-2}\frac{\partial}{\partial \phi_{m-4}} + \cdots + m\phi_{-m+2}\frac{\partial}{\partial \phi_{-m}} - f_{-1}\frac{\partial}{\partial f_{+1}}\right] T_i. \tag{14}$$

Moreover, when the induced transformations are applied to both members of equation (10), F and F_i $(i = 1, \cdots, s)$ remain unaltered since these functions reduce to ordinary covariants in the domain $R(1, 0, 0)$. Then this result of transforming (10) takes the form

$$F = T'_1 F_1 + T'_2 F_2 + \cdots + T'_s F_s.$$

Therefore, unless the quantity in brackets in (14) vanishes, this expression for F is one of the other possible representations of F as a linear function of F_i $(i = 1, \cdots, s)$ constructed upon the principle that the set of auxiliary factors T_i is not unique (cf. proposition (*a*)). But this is impossible by (*B*), for, unless the bracket in (14) is zero, T'_i is not isobaric. Hence $T'_i = T_i$ and the property of seminvariance is established.

Conclusion. Thus, by the principle of correspondence, in accordance with proposition (*d*), we have established the relations

$$T_i(\phi_m, \phi_{m-2}, \cdots; f_{+1}, f_{-1}) = R^\lambda\, T_i(a_0, a_1, \cdots; x_1, x_2) \qquad (i = 1, \cdots, s), \tag{15}$$

where R is the determinant of τ. We can now prove that $\lambda = 0$, so that T_i is absolutely equal to a concomitant belonging to the domain $R(1, 0, 0)$. This is done by showing that T_i is of degree zero in the parameters α_1, α_2, β_0, β_1, a fact which is fairly evident from empirical considerations, but an analytic proof of which is not without interest.

Note that the T-s, though fractional in the parameters are homogeneous in them, and hence, by Euler's theorem,

$$\nabla T_i = \left(\alpha_1 \frac{\partial}{\partial \alpha_1} + \alpha_2 \frac{\partial}{\partial \alpha_2} + \beta_0 \frac{\partial}{\partial \beta_0} + \beta_1 \frac{\partial}{\partial \beta_1}\right) T_i = t_i' \, T_i,$$

where t_i' is the degree of T_i in the parameters. Now suppose that

$$t_1' = t_2' = \cdots = t_r'$$

and operate by ∇ upon both members of (10). Since the F-s are free from α_1, α_2, β_0, β_1, the result is

$$t_1' (T_1 F_1 + \cdots + T_r F_r) + \cdots + t_s' F_s = 0;$$

or

$$t_1 P_1 + t_2 P_2 + \cdots + T_v P_v = 0, \tag{16}$$

where the degrees $t_1, \cdots, t_v$ are all different and $v \geqq 1$. Next operate $v - 1$ times in succession by ∇ upon (16), and we get the linear system

$$t_1^k P_1 + t_2^k P_2 + \cdots + t_v^k P_v = 0 \qquad (k = 1, \cdots, v).$$

The determinant C of this linear system is the circulant of the v distinct numbers t_i, i. e.,

$$C = \begin{vmatrix} t_1 & t_2 & \cdots & t_v \\ t_1^2 & t_2^2 & \cdots & t_v^2 \\ \cdot & \cdot & \cdot & \cdot \\ t_1^v & t_2^v & & t_v^v \end{vmatrix} \neq 0.$$

We would therefore have all P-s zero, and hence the contradiction $F = 0$, against which eventuality there are only two alternatives, viz., first, that the t-s should all be zero due to the fact that the numerator and denominator of each function T_i are of the same degree in the parameters, or, secondly, that the T-s are free from the parameters. Thus $\lambda = 0$, and (15) shows that the T_i are concomitants belonging to $R(1, 0, 0)$. This completes the proof of the theorem of finiteness.

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ON THE THEORY OF DEVELOPMENTS OF AN ABSTRACT CLASS IN RELATION TO THE CALCUL FONCTIONNEL*

BY

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In his *Introduction to a Form of General Analysis*† E. H. Moore has called attention to the great importance of developments Δ in analysis, and has used them in a general theory which includes the theories of continuous functions and convergent series. The authors of the present paper have made further studies of the properties of developments in relation to the theory of Moore.‡

It is the purpose of the present paper to develop the theory of developments along lines inaugurated by Fréchet and developed by him and other investigators. The methods of analysis suggested by this theory have led the authors to results in the Calcul Fonctionnel, some of which have been published previously.§

The theory of developments Δ is placed, in the present paper, into close relation with the theories of systems $(\mathfrak{L})$ and $(\mathfrak{B})$ of Fréchet‖ and the topological space of Hausdorff.¶

We develop the general theory in terms of five completely independent properties of a development Δ which together suffice to make the developed class $\mathfrak{P}$ a compact metric space (cf. Hausdorff, loc. cit.).

The theory is applied to determine necessary and sufficient conditions that a topological space be a compact metric space. A further application is made to spaces s satisfying axiom systems Σ_1 or Σ_2 introduced by R. L.

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† *The New Haven Mathematical Colloquium* (Yale University Press, New Haven, 1910), pp. 1–150.

‡ A. D. Pitcher, *Interrelations of eight fundamental properties of classes of functions*, Kansas University Science Bulletin, vol. 7, no. 1 (1914), pp. 1–67; and E. W. Chittenden, *Infinite developments and the composition property* $(K_{12}B_1)_*$ *in General Analysis*, Rendiconti di Palermo, vol. 39 (1915), pp. 81–108.

§ A. D. Pitcher and E. W. Chittenden, *On the foundations of the calcul fonctionnel of Fréchet*, these Transactions, vol. 19 (1918), pp. 66–78.

‖ Cf. *Sur quelques points du calcul fonctionnel*, Rendiconti di Palermo, vol. 22 (1906), p. 1–64.

¶ *Grundzüge der Mengenlehre* (Veit and Co., Leipzig, 1914), p. 213.

Moore as bases for a theory of plane curves in non-metrical analysis situs.* It is shown that each such space s is the sum of an enumerable set of compact metric sets. A development Δ of s is defined such that the associated distance function δ is equivalent to an écart with respect to limit of a sequence.

1. Let $a = x_1 < x_2 < \cdots < x_m = b$ denote a set of points from an interval (a, b). The corresponding set of intervals (x_i, x_{i+1}), $i = 1, 2, 3, \cdots n-1$, is a division of the interval (a, b) of norm δ, where δ is the greatest value of $x_{i+1} - x_i$. Consider a sequence of divisions of the interval (a, b) with norms, $\delta_1, \delta_2, \cdots$ decreasing to the limit zero. In terms of such a sequence of divisions one may define the Riemann integral $\int_a^b f(x)\,dx$. Denote by $x_1^m, x_2^m, \cdots x_{n_m}^m$ the division points corresponding to the norm δ_m, and by $\mathfrak{P}^{mn}$ the interval (x_n^m, x_{n+1}^m). The system $\Delta \equiv ((\mathfrak{P}^{mn}))$ of all such intervals $\mathfrak{P}^{mn}$ is an example of a development of the interval (a, b).†

In general a development Δ of an abstract class $\mathfrak{P}$ is a sequence of systems Δ^m, called *stages*, each system Δ^m being composed of subclasses $\mathfrak{P}^{ml}$ of $\mathfrak{P}$, the index l having for each integer m a range $\mathfrak{L}^m$.‡ In the present paper we restrict ourselves to the cases in which the classes $\mathfrak{L}^m$ are composed of integers in the natural order and the classes $\mathfrak{P}^{ml}$ are existent classes for every value of the composite index ml in the system $((ml))$ of indices for the development Δ.

We make here the fundamental hypothesis that we are given a class $\mathfrak{P}$ and a development Δ of $\mathfrak{P}$.

2. M. Fréchet has recently introduced the following considerations: a class of elements is a class (V) if to every element p of the classt here is assigned a family of sets V_p called *neighborhoods*§ of p.

Denote by V_p^m the class of all elements of $\mathfrak{P}$ which belong to a class $\mathfrak{P}^{m'l}$ $(m' \geqq m)$ containing p. This class V_p^m will be called *the neighborhood of rank* m of p defined by the development Δ. Then $\mathfrak{P}$ is a class (V). If a class V contains V_p^m and is a proper subclass of V_p^{m-1} it will be called a neighborhood of p of rank m. Evidently such a neighborhood contains all neighborhoods of p of higher rank.

In terms of the neighborhoods V_p^m we define a distance function $\delta(p, q)$ as follows: if q belongs to V_p^m and not to V_p^{m+1} then $\delta(p, q) = 1/m$; if q

* *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), pp. 131–164.

† Many examples of developments are given in the articles of E. H. Moore, Pitcher, and Chittenden cited above. The following example of a development of the class of positive integers ($\mathfrak{P}^{III}$) may be of interest. Denote by $\mathfrak{P}^{m1}$ the class of integers $p \leqq m$, by $\mathfrak{P}^{m2}$ the class of integers $p > m$. Obviously the Cauchy condition for convergence of a sequence $\{a_n\}$ is equivalent to the statement: the oscillation of the sequence on $\mathfrak{P}^{m2}$ approaches zero with m.

‡ E. H. Moore (loc. cit.) assumes that the index l has for each m a finite range. Such developments will be called finite.

§ Comptes Rendus, vol. 165 (1917), p. 359. Fréchet uses the term "voisinages."

does not belong to V_p^m for any m, $\delta(p, q) = 1$; if q belongs to V_p^m for every m, then $\delta(p, q) = 0$; for every p, $\delta(p, p) = 0$.*

For the purposes of this paper we shall define limiting element as follows: *an element p is a limiting element of a class $\mathfrak{Q}$ if and only if every neighborhood of p contains an infinity of elements of $\mathfrak{Q}$.*†

By an obvious application of the Zermelo axiom of choice we obtain the following important proposition:

If p is a limiting element of a class $\mathfrak{Q}$, $\mathfrak{Q}$ contains a sequence $\{q_n\}$ of elements q_n, no two alike, such that $\mathrm{L}_n\, \delta(q_n, p) = 0$, that is, p is a limit of the sequence $\{q_n\}$.

3. The notion of fundamental sequence is helpful in studying developments. A fundamental sequence F is a sequence $\mathrm{F} \equiv \{\mathfrak{P}^{m_n l_n}\}$ of classes of Δ subject to the conditions: $m_n \geqq m_{n-1}$; $\mathrm{L}_n\, m_n = \infty$; *the classes;* $\mathfrak{P}^{m_1 l_1}, \mathfrak{P}^{m_2 l_2}, \cdots, \mathfrak{P}^{m_n l_n}$, *have at least one common element for every value of n.*

A fundamental sequence F of the development Δ is *closed* if there is an element common to all the classes $\mathfrak{P}^{m_n l_n}$. Otherwise F is said to be *open*. The class of all elements belonging to every $\mathfrak{P}^{m_n l_n}$ is called the core of the fundamental sequence and denoted by $\mathfrak{C}(\mathrm{F})$. If an element p is contained in the core of F, then F is said to belong to p and is denoted by F_p.

A sequence $S \equiv \{p_n\}$ is *connected* with a sequence $\{\mathfrak{P}^{m_n l_n}\}$ (which need not be a sequence F) if there exists a sequence $\mathfrak{P}^{\overline{m}_n \overline{l}_n}$ of the classes $\mathfrak{P}^{m_n l_n}$ such that $L_n\, \overline{m}_n = \infty$, and an integer n_0 such that for every $n \geqq n_0$, p_n is contained in $\mathfrak{P}^{\overline{m}_n \overline{l}_n}$.

THEOREM 1. *A necessary and sufficient condition that an element p be a limit of a sequence S is that S be connected with a fundamental sequence F_p.*

Since $\mathrm{L}_n\, \delta(p_n, p) = 0$ there is an integer n_e (for every positive number e) such that if $n \geqq n_e$ then $\delta(p_n, p) \leqq e$. Choose m so that $m + 1 > 1/e \geqq m$. Then p_n must belong to the neighborhood V_p^m. Hence there is a class $\mathfrak{P}^{m_n l_n}$ $(m_n \geqq m)$ such that $\mathfrak{P}^{m_n l_n}$ contains both p and p_n. Since p is common to the classes $\mathfrak{P}^{m_n l_n}$ so obtained they may be arranged to form a sequence F_p with which S is connected by definition.

To prove that the condition is also sufficient we need only observe that if s is connected with a sequence F_p, then for every m there is an n_m such that if $n \geqq n_m$ then $\mathfrak{P}^{m_n l_n}$ is contained in V_p^m. But there exists an integer k_m

* This definition is a special case of a definition of distance in terms of a K relation given by T. H. Hildebrandt, *A contribution to the foundations of Fréchet's calcul fonctionnel*, American Journal of Mathematics, vol. 34 (1912), p. 248.

† This definition is more restrictive than that of Fréchet (loc. cit.) and becomes equivalent to his only when every two neighborhoods of p have a neighborhood of p in common, p has an infinity of distinct neighborhoods, and the neighborhoods of p have no common element, except possibly p. Only the first of these conditions is satisfied by the neighborhoods $\mathfrak{V}_p^m$ in general.

such that p_k $(k \geqq k_m)$ belongs to some class $\mathfrak{P}^{m_n l_n}$ $(n \geqq n_m)$. Therefore p_k belongs to V_p^m and we have $\delta(p_k, p) \leqq 1/m$.

It is important to state the conditions under which no sequence s has more than one limit. We have, as can easily be verified

THEOREM 2. *A necessary and sufficient condition that limit of a sequence* S *be unique is that* Δ *have the following properties:*

if F *is a fundamental sequence the core of* F *contains at most one element;*

if a sequence S *is connected with two closed fundamental sequences* F_1, F_2 *then the cores of* F_1 *and* F_2 *have a common element.*

4. The development Δ is: *finite*, if for every integer m the number of classes $\mathfrak{P}^{ml}$ is finite; *complete*, if for every m and element p there is a class $\mathfrak{P}^{ml}$ containing p; *closed*, if every fundamental sequence F is closed.

THEOREM 3. *If the development* Δ *is complete there exists an enumerable subclass* $\mathfrak{R}$ *of* $\mathfrak{P}$ *such that* $\mathfrak{P} = \mathfrak{R} + \mathfrak{R}'$.

There exists a class $\mathfrak{R} \equiv [r^{ml}/ml]$ of elements of $\mathfrak{P}$ such that r^{ml} belongs to $\mathfrak{P}^{ml}$, and if $\mathfrak{P}^{ml}$, $\mathfrak{P}^{m'l'}$, (where ml and $m'l'$ are distinct) are not subclasses of $\mathfrak{R}$ then r^{ml} is distinct from $r^{m'l'}$.*

S nce Δ is complete there exists for every element p of $\mathfrak{P}$ a fundamental sequence $F_p \equiv \{\mathfrak{P}^{m_n l_n}\}$ of Δ. If p is not an element of $\mathfrak{R}$ the class $[r^{m_n l_n}/n]$ is infinite. The sequence $\{r^{m_n l_n}\}$ is connected with F_p and has the limit p (Theorem 1). It follows that every element p of $\mathfrak{P}$ belongs to $\mathfrak{R} + \mathfrak{R}'$ which was to be proved.

With Hildebrandt (cf. loc. cit., p. 278) we shall say that when $\mathfrak{P} = \mathfrak{R} + \mathfrak{R}'$, $\mathfrak{P}$ is *separable*. If $\mathfrak{P} = \mathfrak{R}'$, $\mathfrak{P}$ is separable in the sense of Fréchet.

LEMMA. *If the development* Δ *is finite and complete every infinite subclass of* $\mathfrak{P}$ *contains an infinite sequence* S *of distinct elements connected with a fundamental sequence* F.

Let $\mathfrak{Q}$ be any infinite subclass of $\mathfrak{P}$. Some class $\mathfrak{P}^{1l_1}$ must contain an infinite subclass $\mathfrak{Q}_1$ of $\mathfrak{Q}$. Likewise some class $\mathfrak{P}^{2l_2}$ must contain an infinite subclass $\mathfrak{Q}_2$ of $\mathfrak{Q}_1$, etc. The sequence $\{\mathfrak{P}^{ml_m}\}$ so obtained is evidently a fundamental sequence F. If the classes $\mathfrak{Q}_m$ have an infinite common subclass it contains the required sequence S. If not then infinitely many of the classes $Q_m - Q_{m+1}$ are non-null, say those with indices m'. Let $q_{m'}$ be an element of $\mathfrak{Q}_{m'} - \mathfrak{Q}_{m'+1}$. Then $S = \{q_{m'}\}$ is the required sequence.

THEOREM 4. *If the development* Δ *is finite, complete, and closed, the class* $\mathfrak{P}$ *is compact and separable.*

* This proposition depends upon the axiom of choice (Zermelo) and the principle of transfinite induction (Cantor). The proof is easy but tedious and is omitted. The argument is based upon the following principle of selection: An index ml precedes an index $m'l'$ if $m < m'$, or if $l < l'$ when $m = m'$. Then r^{ml} is any element of $\mathfrak{P}^{ml}$ which has not previously been selected. If there is no such element in $\mathfrak{P}^{ml}$ then r^{ml} may be any element of $\mathfrak{P}^{ml}$.

A system $R \equiv ((r^{ml}))$ is called by E. H. Moore (loc. cit.) a representative system of a development Δ.

This theorem is an immediate consequence of Theorem 3 and the lemma.

Let $\mu \equiv \mu(p)$ be any real single-valued function defined on $\mathfrak{P}$. Let $\mathfrak{A}_\mu$ be the class of all values $\mu(p)$ assumed by the function μ on $\mathfrak{P}$. If μ is uniformly continuous relative to $\delta(p, q)$ and if Δ is finite and complete it is easy to see that $\mathfrak{A}_\mu$ is bounded. Conversely if $\mathfrak{A}_\mu$ is bounded there exists a finite, complete development of $\mathfrak{P}$ such that μ is uniformly continuous with respect to the corresponding distance function δ. Hence

THEOREM 5. *A necessary and sufficient condition that $\mathfrak{A}_\mu$ be a bounded set is that $\mathfrak{P}$ admit a development Δ such that μ is uniformly continuous with respect to the corresponding distance function δ.*

COROLLARY. *A necessary and sufficient condition that $\mathfrak{A}_\mu$ be compact and closed is that $\mathfrak{P}$ admit a finite complete closed development Δ such that μ is uniformly continuous relative to the corresponding distance function δ.*

The corollary follows readily from Theorems 1 and 4.

5. Let $\mathfrak{Q}$ be a subclass of $\mathfrak{P}$ and $\mathfrak{Q}^{ml}$ be the divisor of $\mathfrak{Q}$ and $\mathfrak{P}^{ml}$. The system $((\mathfrak{Q}^{ml}))$, in which the indices ml of null classes $\mathfrak{Q}^{ml}$ are dropped, is a development $\Delta(\mathfrak{Q})$ obtained by reduction relative to $\mathfrak{Q}$ from the given development Δ.*

Evidently every fundamental sequence F of $\Delta(\mathfrak{Q})$ determines a fundamental sequence F of Δ. Hence any sequence S of elements of $\mathfrak{Q}$ which has a limit relative to $\Delta(\mathfrak{Q})$ has the same limit relative to Δ. Conversely, if a sequence S of $\mathfrak{Q}$ has a limit in $\mathfrak{Q}$ relative to Δ it has the same limit relative to $\Delta(\mathfrak{Q})$. The properties: finite, complete, are easily seen to be invariant under the reduction.

THEOREM 6. *If the development Δ is closed and $\mathfrak{Q}$ is closed then $\Delta(\mathfrak{Q})$ is closed.*

The assumption that $\Delta(\mathfrak{Q})$ contains an open fundamental sequence leads to the conclusion that $\mathfrak{Q}$ has a limiting element in $\mathfrak{P} - \mathfrak{Q}$ contrary to hypothesis.

THEOREM 7. *If $\Delta(\mathfrak{Q})$ is finite, complete, and closed, $\mathfrak{Q}$ is self-compact; that is, every infinite subclass of $\mathfrak{Q}$ has a limiting element in $\mathfrak{Q}$; furthermore $\mathfrak{Q}$ is separable.*

This theorem is a corollary of Theorem 4. The class $\mathfrak{Q}$ need not be closed.

6. An element p is *interior* to a set $\mathfrak{D}$ if $\mathfrak{D}$ contains a neighborhood of p. A class $\mathfrak{D}$ consisting entirely of interior elements will be called a domain.†

THEOREM 8. *If the development Δ is finite, complete, and closed and $\mathfrak{Q}$ is a closed set then any family $(\mathfrak{D})$ of domains whose sum contains $\mathfrak{Q}$, contains a finite subfamily $\mathfrak{D}_1, \mathfrak{D}_2, \cdots, \mathfrak{D}_n$ with the same property.*

* Cf. E. H. Moore, loc. cit., p. 91, also Pitcher, loc. cit., p. 24.

† Hausdorff, loc. cit., p. 215, calls attention to the importance of this concept. He uses the term "Gebiet" in this connection. R. L. Moore, loc. cit., p. 36, following the usage of Weierstrass assumes that a domain is also a connected set.

Let $\mathfrak{Q}$ be a closed set and $(\mathfrak{D})$ any family of domains, whose sum contains $\mathfrak{Q}$, for which the theorem fails. Then $\mathfrak{Q}$ is an infinite set. If $\mathfrak{Q}$ is divided into a finite number of parts $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3, \cdots \mathfrak{Q}_n$, there will be at least one part which will not be contained in any finite subfamily of $(\mathfrak{D})$. Since Δ is finite and complete there is a class $\mathfrak{P}^{1l_1}$ containing an infinite subclass $\mathfrak{Q}_1$ of $\mathfrak{Q}$ which is not contained in the sum class of any finite subfamily of $(\mathfrak{D})$. Likewise there will be a class $\mathfrak{P}^{2l_2}$ containing an infinite subclass $\mathfrak{Q}_2$ of $\mathfrak{Q}_1$ in the same relation to $(\mathfrak{D})$, etc. There exists, therefore, an infinite sequence $\{\mathfrak{Q}_n\}$ of infinite classes, each contained in its predecessor, and a corresponding sequence $F \equiv \{\mathfrak{P}^{ml_m}\}$ which is evidently a fundamental sequence of Δ. Any element of the core of F is easily seen to be a limiting element of $\mathfrak{Q}$ and therefore, since $\mathfrak{Q}$ is closed by hypothesis, is an element of $\mathfrak{Q}$. Let q be an element of the core of F and $\mathfrak{D}_q$ be any domain of $(\mathfrak{D})$ containing q. Then $\mathfrak{D}_q$ contains a neighborhood V_q^m of q which must contain the elements of $\mathfrak{P}^{ml_m}$ and therefore contains $\mathfrak{Q}_m$. This contradicts the hypothesis that $\mathfrak{Q}_m$ is not contained in any finite subfamily of $(\mathfrak{D})$.*

THEOREM 9. *If the development $\Delta(\mathfrak{Q})$ derived by reduction relative to $\mathfrak{Q}$ is finite, complete, and closed, and $(\mathfrak{D})$ is any family of domains whose sum contains $\mathfrak{Q}$, there is a finite subfamily of $(\mathfrak{D})$ whose sum contains $\mathfrak{Q}$.*

This theorem is an evident generalization of Theorem 8. As remarked above the class $\mathfrak{Q}$ need not be closed relative to Δ. The domains $\mathfrak{D}$ need not be domains relative to $\mathfrak{P}$ and Δ.

THEOREM 10. *If the development $\Delta(\mathfrak{Q})$ is finite, complete, and closed, and if the derived class of every subclass of $\mathfrak{Q}$ is closed, then every family $(\mathfrak{H})$ of classes*

* The theorem fails for more general families $(\mathfrak{D})$ as is shown by the following example. Let $\mathfrak{P}$ denote the interval (0, 1). We define a development Δ of $\mathfrak{P}$. Let $\mathfrak{P}^{ml}$ be the interval

$$\left(\frac{l-1}{m}, \frac{l}{m}\right), \qquad (l = 1, 2, 3, \cdots m),$$

with the exception, each class which contains $1 - 1/m$ is to contain the element $1/m$ $(m > 2)$. This development is finite complete and closed, hence $\mathfrak{P}$ admits the property of Theorem 8. But each element of $\mathfrak{P}$ is interior to some one of the following set of classes:

$$\mathfrak{D}_0 \equiv (0, \tfrac{1}{2}); \quad D_1 \equiv (\tfrac{1}{4}, \tfrac{3}{4}); \quad D_2 \equiv (\tfrac{1}{2}, 1);$$

$$\mathfrak{D}_m \equiv \left[\frac{1}{m} - \frac{1}{2m} < p < \frac{1}{m} + \frac{1}{2m}\right] + \left[1 - \frac{1}{m} - \frac{1}{2m} < p < 1 - \frac{1}{m} + \frac{1}{2m}\right],$$

$$m = 3, 4, 5, \cdots,$$

$$\mathfrak{D}_\omega \equiv [\tfrac{1}{3}, \tfrac{1}{4}, \cdots] + [\tfrac{2}{3} \leqq p \leqq 1].$$

The elements $\frac{1}{3}, \frac{1}{4}, \cdots$ are not covered by any finite subfamily of $(\mathfrak{D})$ although for every m, $1/m$ is interior to $\mathfrak{D}_m$. That is the more general form of the Heine-Borel property does not hold. It will be noticed that the class $\mathfrak{D}_\omega$ is not a domain. Let $\mathfrak{Q}$ denote the set of irrational points of the interval $(0, \frac{1}{2})$. Then $\mathfrak{Q}' = (0, \frac{1}{2})$, while $\mathfrak{Q}''$ contains the element $p = 1$. That is in the system just defined, derived classes are not in general closed (cf. Theorem 10).

$\mathfrak{H}$, such that each element of $\mathfrak{Q}$ is INTERIOR *to some class $\mathfrak{H}$, contains a finite subfamily with the same property.**

Exactly as in the proof of Theorem 8 we obtain a sequence $\{\mathfrak{Q}_m\}$ and a fundamental sequence F of $\Delta(\mathfrak{Q})$. Let q be an element of the core of F. Then q is interior to some class $\mathfrak{H}_q$, which must as before, contain a class $\mathfrak{Q}_m$, for sufficiently large m. We shall prove that for some value of $m' > m$, $\mathfrak{Q}_{m'}$ is interior to $\mathfrak{H}_q$. If there is no such value of m', then for every $m' > m$, $\mathfrak{Q}_{m'}$ contains an element $q_{m'}$ not interior to $\mathfrak{H}_q$. The sequence $\{q_{m'} > m\}$ so obtained has the limit q. We may suppose the sequence $\{q_{m'}\}$ so chosen that it contains infinitely many distinct elements. It follows from a theorem of Hedrick (these Transactions, vol. 12 (1911), p. 286) on domains in which derived classes are closed that at least one of the elements $q_{m'}$ is interior to $\mathfrak{H}_q$. This is the desired contradiction.

As an immediate consequence of Theorem 10 we have

THEOREM 11. *If the development Δ is finite, complete, and closed, and if the derived class of every class is closed, then for every continuous function μ and small positive number e there exists a set of elements $p_1, p_2, p_3, \cdots, p_n$, and a corresponding set of integers $m_1, m_2, \cdots, m_n$ such that every element p is interior to some $\mathfrak{V}_{p_j}^{m_j}$ on which the oscillation of μ is less than e.*

7. The development Δ is *regular* if for every pair of elements (p_1, p_2) contained in a class $\mathfrak{P}^{ml}$ $(m > 1)$ there is a class $\mathfrak{P}^{m-1,\,l}$ which also contains the pair (p_1, p_2).

Two sequences $S_1 \equiv \{p_{1n}\}$, $S_2 \equiv \{p_{2n}\}$ are connected in case

$$L_n\, \delta(p_{1n}, p_{2n}) = 0.$$

A necessary and sufficient condition that two sequences S_1, S_2 be connected is that for every value of n there is a class $\mathfrak{P}^{m_n l_n}$ containing p_{1n} and p_{2n}, where $L_n\, m_n = \infty$.

THEOREM 12. *If the development Δ is finite, complete, and regular, then for every pair of connected sequences S_1, S_2 there exists a pair S_1', S_2' of connected subsequences of S_1, S_2, respectively, and a fundamental sequence F with which both S_1' and S_2' are connected.*

Since Δ is regular and S_1, S_2 are connected each pair (p_{1n}, p_{2n}) is contained in some class $\mathfrak{P}^{1l}$. Hence an infinity $\mathfrak{J}_1$ of these pairs must lie in a single class $\mathfrak{P}^{1l_1}$. Similarly (because $L_n\, m_n = \infty$ in the condition above) an infinite subclass $\mathfrak{J}_2$ of $\mathfrak{J}_1$ is contained in a single class $\mathfrak{P}^{2l_2}$, etc. We thus obtain a fundamental sequence $F \equiv \{\mathfrak{P}^{ml_m}\}$ of Δ which is the fundamental sequence of the theorem. Let (p_{1n_1}, p_{2n_1}) be the first element of $\mathfrak{J}_1$ which appears in the sequence $\{(p_{1n}, p_{2n})\}$. Let (p_{1n_2}, p_{2n_2}) be the first element of $\mathfrak{J}_2$ to follow (p_{1n_1}, p_{2n_1}) in the same sequence, etc. The sequences $S_1' \equiv \{p_{1n_k}\}$, $S_2' \equiv \{p_{2n_k}\}$ so obtained are connected with each other and with F.

* Theorems 9 and 10 are equivalent if limit is unique. Cf. Theorem 21.

If the fundamental sequence F is closed, S_1', S_2' have a common limit. Hence

THEOREM 13. *If the development Δ is finite, complete, closed, and regular the system $(\mathfrak{P}; \Delta)$ is biextremal.**

From Theorem 13 and Theorems 11 and 12 of our previous paper, and the further fact that a continuous function on a closed set attains its bounds one readily sees that:

THEOREM 14. *If $\Delta(\Omega)$ is finite, complete, closed, and regular, then any function μ continuous on Ω, is uniformly continuous on Ω, bounded and attains its bounds. Furthermore if Ω is a connected† set the function μ attains on Ω every value between its bounds.*

8. The theory of coherent systems $(\mathfrak{P}; \delta)$ was developed in the previous paper. A system $(\mathfrak{P}; \delta)$ is *coherent* if whenever two sequences S_1, S_2 are *connected* (cf. § 7) then every limit of S_1 is a limit of S_2. The following theorem is a consequence of Theorem 4 above and Theorem 6 of the previous paper.

THEOREM 15. *If the system $(\mathfrak{P}, \delta)$ derived from the development Δ of the given class $\mathfrak{P}$ is coherent, the derived class of every class is closed; and if furthermore Δ is finite, complete, and closed, δ is equivalent with respect to limit to a distance function $\bar{\delta}^{2345}$.*‡

The relation of Theorem 15 to Theorem 10 should be noted.

THEOREM 16. *Under the hypothesis of Theorem 15, every continuous function μ is uniformly continuous, bounded, and attains its bounds.*

This theorem is to be compared with Theorem 5. We will show that μ is uniformly continuous. Under the contrary hypothesis there exists a positive number e_0 and for every n a pair of elements p_{1n}, p_{2n} such that

$$\delta(p_{1n}, p_{2n}) \leqq 1/n,$$

while

$$|\mu(p_{1n}) - \mu(p_{2n})| > e_0.$$

The sequences $S_1 = \{p_{1n}\}$, $S_2 = \{p_{2n}\}$ are connected. If S_1 contains an element p repeated infinitely often, then p is a limit of S_2, which leads at once to a contradiction. If the elements of S_1 are not repeated then S_1 contains infinitely many elements and, since $\mathfrak{P}$ is compact, contains a subsequence S_1' with a limit p. Since the corresponding sequence S_2' must, because of the definition of coherence, have the limit p, the inequality assumed contradicts the hypothesis of continuity.

* Cf. Pitcher-Chittenden, loc. cit., Section 4. A system is *biextremal* in case there exists for every pair of connected sequences S_1, S_2 a pair S_1', S_2' of connected subsequences of S_1, S_2, respectively, which have a common limit.

† Cf. previous paper, p. 68, for definition of this term.

‡ The properties 2, 3, 4, 5 of a distance function δ are defined in Section 1. of the previous paper. If $\bar{\delta}$ has the further property δ^1; $\delta(p, q) = 0$ implies $p = q$, then $\bar{\delta}$ is a voisinage as defined in the thesis of Fréchet (loc. cit.).

THEOREM 17. *Every coherent system derived from a system* $(\mathfrak{P}; \Delta)$ *implies a coherent system such that to every element and limit relation in the first there corresponds a unique element and limit relation in the second, and in the second system limit is unique.**

Denote by $\mathfrak{C}_p$ the class of all elements q satisfying the equation $\delta(p, q) = 0$. If two classes $\mathfrak{C}_p$ and $\mathfrak{C}_q$ have a common element they are identical. For the identical sequences corresponding to the elements p and q, respectively, are connected with the identical sequence of this common element, which has the limit q. It follows from the definition of coherence that $\delta(p, q) = 0$.

Let $\mathfrak{S} \equiv [s]$ be a definite subclass of $\mathfrak{P}$ such that for every s there is a class $\mathfrak{C}$ containing s, and for every class $\mathfrak{C}$ there is a unique element s of $\mathfrak{S}$ contained in $\mathfrak{C}$ (Zermelo Axiom).

Let $\mathfrak{W} \equiv [w]$ denote any class in 1–1 correspondence with the class of all classes $\mathfrak{C}$. We define $\delta(w_1, w_2) = \delta(s_1, s_2)$, where s_1, s_2 correspond to w_1, w_2, respectively. To each element of p corresponds a unique pair of elements s, w. If w_n corresponds to p_n and $\mathrm{L}_n\, \delta(p_n, p) = 0$, then $\mathrm{L}_n\, \delta(w_n, w) = 0$. In the system $(\mathfrak{W}; \delta)$ limit is unique. For every sequence $\{w_n\}$ with limit w, there is a unique sequence $\{s_n\}$ with corresponding limits s. Hence, if $\{w_n\}$ has limits w', w'', the sequence $\{s_n\}$ has corresponding limits s', s''. But s', s'' must belong to the same class $\mathfrak{C}$. Therefore $s' = s''$, $w' = w''$, which was to be proved. A similar argument shows that $(\mathfrak{W}; \delta)$ is a coherent system.

COROLLARY: *If* μ *is continuous on* $\mathfrak{P}$ *and* $\mathfrak{W}$ *corresponds to* $\mathfrak{P}$, *the function* $\mu(w) = \mu(s)$ *is continuous on* $\mathfrak{W}$ *and assumes every value on* $\mathfrak{W}$ *that it assumes on* $\mathfrak{P}$.†

9. Denote by $\mathfrak{R}_p^m$ the class of all elements q such that p and q are contained in some class $\mathfrak{P}^{m'l'}$ for every $m' \leqq m$. Evidently $\mathfrak{R}_p^m$ contains $\mathfrak{R}_p^{m+1}$.

THEOREM 18. *If the development* Δ *is finite, complete, and closed, and if limit of a sequence is unique, then every class* $\mathfrak{R}_p^m$ *contains an infinite subclass of every class* $\mathfrak{Q}$ *which has* p *for a limiting element.*

COROLLARY. *The class* $\mathfrak{R}_p^m$ *contains all but a finite number of elements of any sequence with the limit* p.

The class $\mathfrak{Q}$ contains an infinite sequence $\mathrm{S} = \{q_n\}$ with the limit p, containing an infinity of distinct elements. Let $\mathfrak{S}$ denote the class of elements in the sequence S. By means of an argument applied several times before we obtain a sequence $\{\mathfrak{S}_m\}$ of infinite subclasses of $\mathfrak{S}$ each contained in its predecessor and in every class $\mathfrak{P}^{m'l_{m'}}$ $(m' \leqq m)$ of a fundamental sequence F $(m = 1, 2, 3, \cdots)$. Since F is closed and limit is unique, the core of F consists of the element p. Therefore the infinite class $\mathfrak{S}_m$ belongs to $\mathfrak{R}_p^m$ for every value of m.

* Theorem 17 holds for any coherent system $(\mathfrak{P}\ ;\ \delta)$ such that for every element p of $\mathfrak{P}$, $\delta(p, p) = 0$.

† Every continuous function on $\mathfrak{P}$ is a constant on $\mathfrak{C}p$ for every p.

THEOREM 19. *Under the hypothesis of Theorem* 18 *the class* $\mathfrak{P}$ *admits a definition of a distance function* ρ, *which is equivalent to* δ *with respect to limit of a sequence, such that the system* $(\mathfrak{P}; \rho)$ *is coherent.**

We define the function $\rho(p, q)$ as follows: if q belongs to R_p^m and does not belong to $\mathfrak{R}_p^{m+1}$ then $\rho(p, q) = 1/m$; if $q = p$, or if q belongs to $\mathfrak{R}_p^m$ for every integer m, then $\rho(p, q) = 0$. Evidently $\delta(p, q) \leqq \rho(p, q)$. Hence every limit in terms of ρ implies a limit in terms of δ. Conversely, suppose that $L_n \delta(p_n, p) = 0$. We must prove that $L_n \rho(p_n, p) = 0$. From the corollary above, all but the first k_m of the elements p_n belong to $\mathfrak{R}_p^m$. Hence if $n > k_m$, $\rho(p_n, p) \leqq 1/m$, which implies $L_n \rho(p_n, p) = 0$. This establishes the equivalence of ρ and δ with respect to limit. We shall now prove that the system $(\mathfrak{P}, \rho)$ is coherent. We have to prove that if $S_1 \equiv \{p_n\}$, $S_2 \equiv \{q_n\}$ satisfy the condition

$$L_n \rho(p_n, q_n) = 0,$$

then any limit of S_1 is a limit of S_2. Suppose that p is the limit of S_1 and is not a limit of S_2. Then S_2 must contain a subsequence $S_2' \equiv \{q_n'\}$ such that for some fixed integer $\overline{m}$, and for every integer n, q_n' does not belong to $\mathfrak{R}_p^{\overline{m}}$. If p_n' is the element of S_1 which corresponds to q_n', we have $L_n \rho(p_n', q_n') = 0$. Hence an infinity $\mathfrak{I}_1$ of the pairs (p_n', q_n') lie in some class $\mathfrak{P}^{1/i_1}$, an infinity of these $\mathfrak{I}_2$ lie in a class $\mathfrak{P}^{2/i_2}$, etc. The fundamental sequence F of Δ so obtained is closed by hypothesis and hence its core consists of the element p. Therefore p is a limit of a subsequence of S_2', which is the desired contradiction.

THEOREM 20. *Under the hypothesis of Theorem* 18 *the distance function* ρ *is equivalent (with respect to limit of a sequence) to an écart.*

Cf. Theorem 19 above and Theorem 7 of the previous paper. As an immediate consequence of preceding theorems we have:

THEOREM 21. *If the development* Δ *is finite, complete, and closed, and if limit of a sequence is unique the class* $\mathfrak{P}$ *is a compact metric space.*†

10. An element p is *singly developed* by a development Δ if for every integer m there exists an integer $m' > m$ such that some class $\mathfrak{P}^{m'/r}$ consists of the single element p. A development Δ is *asingular* in case no element of $\mathfrak{P}$ is singly developed.

THEOREM 22. *If the development* Δ *is finite, complete, asingular and every fundamental sequence of* Δ *has a singular core, then* $\mathfrak{P}$ *is compact, perfect, and separable.*

We will show that $\mathfrak{P} = \mathfrak{R}'$ where $\mathfrak{R}$ is the enumerable class introduced in the proof of Theorem 3. Let $F = \{\mathfrak{P}^{m_n/s_n}\}$ be a fundamental sequence belonging to p. Since Δ is asingular every class $\mathfrak{P}^{m_n/s_n}$ of F contains at least

* If Δ is regular the functions δ and ρ are identical.

† Cf. Hausdorff, loc. cit., p. 211 et seq., for consequences of this theorem.

one element distinct from p. The class of such elements is infinite, since the core of F is singular. Hence from the definition of $\mathfrak{R} \equiv ((r^{ml}))$, the sequence $\mathfrak{S} = \{r^{m_n l_n}\}$ contains infinitely many distinct elements. Since $\mathfrak{S}$ has the limit p, p is in $\mathfrak{R}'$ which was to be proved.

THEOREM 23. *A necessary and sufficient condition that a class* $\mathfrak{P}$ *in a system* $(\mathfrak{L})$ *of Fréchet be a compact class* "(V) NORMALE"* *is that* $\mathfrak{P}$ *admit a finite, complete, csingular, closed development* Δ, *such that limit in* Δ *is equivalent to limit in the system* $(\mathfrak{L})$.

The condition is sufficient. In the system $(\mathfrak{L})$ limit is unique. Hence from Theorem 22, $\mathfrak{P}$ is compact, separable, and perfect. From Theorem 21 it follows that $\mathfrak{P}$ admits a generalization of the theorem of Cauchy.

We show that the condition is necessary by defining a development Δ with the desired properties. Let (p, q) denote the voisinage of elements p and q. Denote by V_p^m the neighborhood of p of rank m. Then p is interior to V_p^m. From the Heine-Borel theorem, which applies to this case, we have, for every positive integer m, a finite set:

$$r_1^m, r_2^m, \cdots r_l^m,$$

of elements such that every element of $\mathfrak{P}$ is interior to at least one of the classes $V_{r_l}^m$. Denote by $\mathfrak{P}^{ml}$ the derived class of $V_{r_l}^m$. The development $\Delta \equiv ((\mathfrak{P}^{ml}))$ is obviously finite, complete, and asingular. The proofs of closure and equivalence can be carried through easily by reference to the properties of voisinage and fundamental sequences.

11. In the foregoing sections the principal results have been stated for closed developments. In the present section we extend these results to a class of open developments by the introduction of ideal elements defined in terms of fundamental sequences.

Two fundamental sequences F_1, F_2 of Δ are connected if there exists an infinite sequence of elements of $\mathfrak{P}$ connected with both F_1 and F_2. It is evident that *if* F_1 *and* F_2 *are fundamental subsequences formed from the classes* $\mathfrak{P}^{m_n l_n}$ *of a fundamental sequence* F *they are connected.*

Two fundamental sequences F and F′ are *connected* of *order* n if there exists a chain, $F_1, F_2, \cdots F_{n-1}$, of $n-1$ fundamental sequences such that F_1 is connected with F, F_2 is connected with F_1, etc., F_{n-1} being connected with F′.

Every fundamental sequence F determines a *family* $\mathfrak{F}$ of fundamental sequences, the family of all fundamental sequences connected with F of order n for some value of n. *It is evident that if* F_1 *and* F_2 *are connected of order* n *they determine the same family* $\mathfrak{F}'$.

The classes $\mathfrak{P}^{ml}$ which belong to the fundamental sequences of a family $\mathfrak{F}$

* A "class (V) normale" is separable, perfect, admits a definition of voisinage and a generalization of the theorem of Cauchy.

form a definite *set* of classes of Δ which we shall also denote by $\mathfrak{F}$, speaking of the *set* $\mathfrak{F}$ of classes of Δ associated with a fundamental sequence F.

If an element p of $\mathfrak{P}$ belongs to the core of a fundamental sequence F of Δ the associated *family* $\mathfrak{F}$ or *set* $\mathfrak{F}$ will be denoted by $\mathfrak{F}_p$. Let $\mathfrak{F}_0$ denote the class of all elements p with which a given family $\mathfrak{F}$ is associated. If $\mathfrak{F}_0$ is a null-class every fundamental sequence of $\mathfrak{F}$ is open.

Let $[\mathfrak{F}]_*$ be the class of all families $\mathfrak{F}$ of Δ which consist entirely of open sequences, and let $\mathfrak{E}_*$ be any set of elements which has no element in common with $\mathfrak{P}$ and is in 1–1 correspondence with $[\mathfrak{F}]_*$. Let p_* be the element of $\mathfrak{E}_*$ which corresponds to a family $\mathfrak{F}$ of $[\mathfrak{F}]_*$. Denote by $\mathfrak{F}_{*0}$: the class $\mathfrak{F}_0$, when $\mathfrak{F}_0$ is not null; and the class of the single element p_*, when $\mathfrak{F}_0$ is a null-class.

We proceed to define a development Δ_* of the class $\mathfrak{P}_* \equiv \mathfrak{P} + \mathfrak{E}_*$ which we term the $*$-extension of Δ. For every class $\mathfrak{P}^{ml}$ there is a definite class $\mathfrak{T}^{ml}$ (which may be a null-class) of sets $\mathfrak{F}$ which contain $\mathfrak{P}^{ml}$. Let $\mathfrak{E}_*^{ml}$ denote the class of all elements contained in some class $\mathfrak{F}_{*0}$ derived from a set $\mathfrak{F}$ of $\mathfrak{T}^{ml}$. The development Δ_* of $\mathfrak{P}_*$ is the system $((\mathfrak{P}_*^{ml}))$ where $\mathfrak{P}_*^{ml} \equiv \mathfrak{P}^{ml} + \mathfrak{E}_*^{ml}(ml)$.

During the remaining part of the discussion in this section of the paper we shall denote by $\{\mathfrak{P}_n\}$ *a sequence of classes of* Δ *such that* $\mathfrak{P}_n = \mathfrak{P}^{m_n l_n}$, *where* $L_n m_n = \infty$ *and* $m_n \leqq m_{n+1}(n)$.

If $F \equiv \{\mathfrak{P}_n\}$ *is a fundamental sequence of* Δ, $F_* \equiv \{\mathfrak{P}_{*n}\}$ *is a closed fundamental sequence of* Δ_*. *The sequence* F_* is called the $*$-extension of F.

Every limit relation of the system $(\mathfrak{P}; \Delta)$ is present in the system $(\mathfrak{P}_*; \Delta_*)$.

We give an example showing that the development Δ_* is not always closed. Let $\mathfrak{P}$ be the class of all real numbers; $\mathfrak{L}^m \equiv [-m, -m+1, \cdots, -2, -1, 1, 2, 3, \cdots]$; $\mathfrak{P}^{ml} \equiv [\text{all } p(>-l) \text{ of the form } ae^{-l}$, where a is a rational number and $e = 2.71828\cdots]$ $(l = -m, -m+1, \cdots, -2, -1)$; $\mathfrak{P}^{ml} \equiv [l - 1/m \leqq p < l]$ $(l = 1, 2, 3, \cdots)$; $\mathfrak{E}_* \equiv [1, 2, 3, \cdots]$. The sequence $\{\mathfrak{P}_*^{m,-m}\}$ is an open fundamental sequence of Δ_*.

Let $\{\mathfrak{T}_n\}$ denote a sequence of sets of families $\mathfrak{F}$ of Δ such that $\mathfrak{T}_n$ contains $\mathfrak{T}_{n+1}(n)$. Let $H \equiv \{\mathfrak{P}_n\}$ satisfy the condition: $\mathfrak{P}_n$ belongs to some sequence of every family $\mathfrak{F}$ of $\mathfrak{T}_n(n)$. The development Δ is *closeable* if every such sequence H of Δ is a fundamental sequence.

The following chain of propositions follow from the hypothesis: *the development* Δ *is closeable.*

Every sequence $\{\mathfrak{P}_n\}$ *formed from the classes of a single set* $\mathfrak{F}$ *of* Δ *is a fundamental sequence.*† This follows from the definition of the closeable property on setting $\mathfrak{T}_n = \mathfrak{F}(n)$. This proposition has the corollary: *any sequence*

† Hence, if Δ is finite the classes of any *set* $\mathfrak{F}$ associated with a fundamental sequence F form a fundamental sequence of Δ of which F is a subsequence.

$\{\mathfrak{P}_n\}$ formed from the classes of two connected fundamental sequences is a fundamental sequence.

If a fundamental sequence is connected with two fundamental sequences they are connected. For the three sequences can be combined to form a single fundamental sequence, and any pair of fundamental subsequences of a fundamental sequence is connected. Hence, *if two fundamental sequences are connected of order n, they are connected of the first order, and every family $\mathfrak{F}$ of fundamental sequences of Δ consists of a fundamental sequence* F *and all sequences connected with* F.

If $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are classes of a set $\mathfrak{F}$ of Δ they have a common element. For $\mathfrak{P}_1$ and $\mathfrak{P}_2$ form part of a sequence $\{\mathfrak{P}_n\}$ of classes of $\mathfrak{F}$ and this sequence must be a fundamental sequence.

If H_* is a fundamental sequence of Δ_*, H_* is the $*$-extension of a fundamental sequence H *of* Δ. Suppose that $H_* \equiv \{\mathfrak{P}_{*n}\}$, and that $H = \{\mathfrak{P}_n\}$ is not a fundamental sequence of Δ. Let n_0 be an integer such that $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_{n_0}$, have no common element. Since $\mathfrak{P}_{*1}, \mathfrak{P}_{*2}, \mathfrak{P}_{*3}, \cdots, \mathfrak{P}_{*n_0}$, have a common element p_*, there is an integer $i \leqq n_0$ such that $\mathfrak{P}_i$ does not contain p_*. The element p_* is either an element of some class $\mathfrak{F}_0$ or else an element of $\mathfrak{E}_*$. In either case p_* determines a family $\mathfrak{F}p_*$, and every class $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_{n_0}$ belongs to the corresponding set $\mathfrak{F}_{p*}$ of classes of Δ. Let $\mathfrak{H}_{*n}$ be the class of all such elements p_* which are common to $\mathfrak{P}_{*1}, \cdots, \mathfrak{P}_{*n}$. Let $\mathfrak{T}_n$ be the class of all families $\mathfrak{F}$ which correspond to elements of $\mathfrak{H}_{*n}$. Then $\mathfrak{T}_n$ contains $\mathfrak{T}_{n+1}$ and $\mathfrak{P}_n$ belongs to every set $\mathfrak{F}$ of $\mathfrak{T}_n$. Since Δ is closeable this contradicts the hypothesis that H is not a fundamental sequence of Δ.

From the definition of $*$-extension and the proposition just proved we obtain the theorem:

THEOREM 24. *If the development Δ is closeable the development Δ_* of $\mathfrak{P}_*$ is closed.*

THEOREM 25. *If Δ is closeable Δ_* is closeable.*

Suppose $\{\mathfrak{T}_{*n}\}$ is a sequence of sets of families $\mathfrak{F}_*$ of fundamental sequences of Δ_*. Then $\mathfrak{T}_{*n}$ is the extension of a set $\mathfrak{T}_n$ of families of Δ, and if $\mathfrak{P}_*^{ml}$ belongs to every family of $\mathfrak{T}_{*n}$ then $\mathfrak{P}^{ml}$ belongs to every family of $\mathfrak{T}_n$.† Hence, if $\{\mathfrak{P}_{*n}\}$ is a sequence of classes of Δ_* such that $\mathfrak{P}_{*n}$ belongs to every family of $\mathfrak{T}_{*n}(n)$, then the class $\mathfrak{P}_n$ corresponding to $\mathfrak{P}_{*n}$ belongs to every family of $\mathfrak{T}_n$. If $\mathfrak{T}_{*n}$ contains $\mathfrak{T}_{*n+1}$, $\mathfrak{T}_n$ contains $\mathfrak{T}_{n+1}(n)$; and it follows from the closeable property of Δ that $\{\mathfrak{P}_n\}$ is a fundamental sequence of Δ. This implies that $\{\mathfrak{P}_{*n}\}$ is a fundamental sequence of Δ_*, which was to be proved.

If Δ is closeable and limit is unique in the system $(\mathfrak{P}; \Delta)$ then every funda-

† This is an immediate consequence of the proposition: if $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are classes of a set $\mathfrak{F}$ they have a common element (together with the definition of Δ_*).

mental sequence of Δ_* has a singular core. Let F_* be a fundamental sequence of Δ_* whose core contains two elements p_*, p'_*. Then p_* and p'_* are associated with the same family F of fundamental sequences of Δ and are, therefore, both elements of $\mathfrak{P}$. It follows from the preceding propositions that p_* and p'_* belong to connected closed fundamental sequences of Δ, and since limit is unique in the system $(\mathfrak{P}; \Delta)$ the cores of these sequences have a common element. As the cores are singular they coincide. This is the desired contradiction. We have

THEOREM 26. *If limit is unique in the system $(\mathfrak{P}; \Delta)$ and if Δ is closeable then limit is unique in the system $(\mathfrak{P}_*, \Delta_*)$ and Δ_* has the closeable property.*

THEOREM 27. *A necessary and sufficient condition that Δ_* be closed is that every sequence $H \equiv \{\mathfrak{P}_n\}$ of Δ with the property:*

(A) there exists a sequence $\{\mathfrak{T}_n\}$ of classes of sets $\mathfrak{F}$ of Δ such that there is no set $\mathfrak{F}$ common to the classes $\mathfrak{T}_n$, $\mathfrak{T}_n$ contains $\mathfrak{T}_{n+1}$, and every set $\mathfrak{F}$ of $\mathfrak{T}_n$ contains $\mathfrak{P}_n$ $(n = 1, 2, 3, \cdots)$;

has the further property:

(B) there exists a family $\mathfrak{F}$ of Δ such that the corresponding set $\mathfrak{F}$ of classes $\mathfrak{P}^{ml}$ contains every class $\mathfrak{P}_n$ of H.

If a sequence H has property *(B)*, then from the definition of $*$-extension H_* is closed. It is easy to see that any open fundamental sequence H_* of Δ_* must correspond to a sequence H of Δ with property *(A)*. This leads at once to a contradiction. Hence the condition is sufficient.

The condition is necessary. The $*$-extension H_* of any sequence H with property *(A)* is a fundamental sequence of Δ. Since Δ_* is closed, H_* is closed. Therefore H belongs to the set F_{p*} where p_* is any element of the core of H_*. Therefore H has property *(B)*, which was to be proved.

THEOREM 28. *If limit is unique in the system $(\mathfrak{P}; \Delta)$ and the development Δ is finite, complete, and closeable, the class $\mathfrak{P}_*$ in the system $(\mathfrak{P}_*, \Delta_*)$ is a compact, metric space.*

THEOREM 29. *If the development Δ is finite, complete, regular, and closeable; the development Δ_* is also finite, complete, regular, and closeable; and the system $(\mathfrak{P}_*; \Delta_*)$ is biextremal. Any function μ continuous on $\mathfrak{P}_*$ is uniformly continuous on $\mathfrak{P}_*$ (and therefore on $\mathfrak{P}$), bounded and attains its bounds.*

Theorems 28 and 29 are consequences of Theorems 13, 14, 21 and the results of the present section.

12 Two developments Δ_1, Δ_2 of the class $\mathfrak{P}$ will, for the purposes of this paper, be said to be *equivalent* if they are equivalent with respect to limit of a sequence.†

† Cf. E. W. Chittenden, On the Equivalence of Relations Kq_1q_2m, American Journal of Mathematics, vol. 39 (1917), p. 266. The necessary and sufficient condition for equivalence of developments is contained in Theorem I of that paper. Two elements p, q are said to be in the relation Kq_1q_2m (determined by Δ) if they both belong to the neighborhood $\mathfrak{V}^m_p$.

It is important to notice that the following operations on a development Δ lead to an equivalent and sometimes simpler development Δ_0.

(1) Any finite number of stages of Δ may be removed. A development which under this operation becomes equivalent to a finite, complete, etc., development is said to be ultimately finite, ultimately complete, etc. All the previous theorems admit of immediate extension to such developments.

(2) If $\mathfrak{P} = \mathfrak{R} + \mathfrak{R}'$, where $\mathfrak{R} = ((r^{ml}))$ is the representative system of a development obtained in § 4, and the derived class of every class is closed; then the development $\Delta_0 \equiv ((\mathfrak{P}_0^{ml}))$, where $\mathfrak{P}_0^{ml} \equiv V^m_{r^{ml}}$, is equivalent to Δ.

(3) If Δ_1, Δ_2 are equivalent, the development Δ whose mth stage contains all the classes of the respective mth stages of Δ_1 and Δ_2 is equivalent to both Δ_1 and Δ_2.

(4) From a given development Δ we may obtain an equivalent *regular* development Δ_0 as follows: if $\mathfrak{P}^{m'l'}$ $(m' > m)$ contains a pair of elements p, q not contained in any class $\mathfrak{P}^{ml}$, adjoin to stage Δ^m of Δ the class $\mathfrak{P}^{m'l'}$. The development Δ_0 thus obtained will in general be infinite. In case Δ_0 is finite and complete the reduction is evidently of importance.

The principle of reduction introduced in § 5 has an important application to incomplete or infinite developments. Denote by $\mathfrak{P}_{-m}$ the class of all elements p which belong to some class $\mathfrak{P}^{m'l}$ for every $m' \geqq m$. Evidently $\mathfrak{P}_{-m}$ is contained in $\mathfrak{P}_{-(m+1)}$.* If Δ is finite the reduction of Δ relative to $\mathfrak{P}_{-m}$ is finite and complete. Hence if Δ is finite and closed the reduced development is finite, complete, and closed. In this case the class $\mathfrak{P}_{-m}$ is self-compact.

The following is an example of a finite closed development relative to which every class $\mathfrak{P}_{-m}$ is compact. Let $\mathfrak{P}$ be the class $[1 \leqq x]$. Let a^{ml} be a set of numbers such that

$$a^{m1} = 1, \qquad 0 < a^{m,\,l+1} - a^{ml} < \frac{1}{2^{m-1}}, \qquad a^{m,\,m2^m} = m + 1.$$

Then $\mathfrak{P}^{ml} \equiv [a^{ml} \leqq x \leqq a^{m,\,l+1}]$ $(l = 1, 2, \cdots, m2^m)$ defines a development of the type indicated. The class $\mathfrak{P}_{-m}$ is the interval $[1 \leqq x \leqq m + 1]$. Such a development can be used to define the improper integral $\int_1^\infty f(x)\,dx$. If the index l^m is not limited as above the development is infinite. Denote by $\mathfrak{Q}_m$ the class of all points contained in classes $\mathfrak{P}^{ml}$ $(l = 1, 2, \cdots l^m)$ where l^m is chosen arbitrarily (in the present case we could choose $l^m = m2^m$, and then $\mathfrak{Q}_m = [1 \leqq x \leqq m + 1]$ as before). Then the reduction relative to $\mathfrak{Q}_m$ is finite, complete, and closed, and $\mathfrak{P}$ is again representable as the sum of a set of compact classes. These examples serve to illustrate the general theory.

The following type of incomplete development is equivalent to a complete development. Suppose that there is a sequence $\{m_\kappa\}$ of integers $m_\kappa (> m_{\kappa-1})$ such that every element of $\mathfrak{P}$ is contained in some class $\mathfrak{P}^{ml}$, $m_\kappa \leqq m < m_{\kappa+1}$,

* The notation is due to E. H. Moore. Cf. Pitcher, loc. cit., pp. 40–42.

for every k. The equivalent complete development Δ_0 is obtained by uniting the stages m_κ to $m_{\kappa+1}$ of Δ, forming stage k of Δ_0.

13. We have developed the theory of developments Δ in terms of five fundamental properties: (1) finite; (2) complete; (3) closed; (4) every fundamental sequence has at most one element in its core; (5) if two closed fundamental sequences are connected with a sequence $\{p_n\}$ their cores have a common element.* We shall show that these five properties are completely independent in the sense of E. H. Moore.† The proof of complete independence requires the exhibition of $2^5 = 32$ examples, representing all possible combinations of the five properties and their negatives.

THEOREM 30. *The five fundamental properties of a development Δ are completely independent.*

The 32 examples are defined in terms of a basis system $(\mathfrak{P}_0, \Delta_0)$ possessing the five properties in question.

The class $\mathfrak{P}_0$ is the interval $[0 \leqq x \leqq 1]$. Each class $\mathfrak{P}^{ml}$ is an interval $[l/2^m \leqq x \leqq (l+1)/2^m]$. $(l^m = 0, 1, 2, \cdots, 2^m - 1)$.

We next define five independent operations on $(\mathfrak{P}_0, \Delta_0)$ each of which removes a specified property of Δ_0.

(1) Adjoin to $\mathfrak{P}$ the points of the open interval $[4 \leqq x]$ and to Δ the classes $\mathfrak{P}^{ml} \equiv [l/2^m \leqq x \leqq (l+1)/2^m]$, $l = (4 \cdot 2^m, 4 \cdot 2^m + 1, \cdots)$.

(2) Remove from Δ all classes $\mathfrak{P}^{ml}$ of index $l = 2^m - 1$.

(3) Remove from $\mathfrak{P}_0$ the point $x = \frac{1}{2}$.

(4) Adjoin to $\mathfrak{P}_0$ the point $x = 2$ and add this point to the classes $\mathfrak{P}^{ml}(m, l = 0)$.

(5) Adjoin to $\mathfrak{P}_0$ the point $x = 3$ and add to Δ the classes $\mathfrak{P}^{ml} \equiv (3, 1/m)$, the index l being assigned for each m the value $l = 2^m$.

A development Δ, of a class $\mathfrak{P}$, having any combination of the five properties is obtainable from $\mathfrak{P}_0$ and Δ_0 by a proper combination of the operations listed above. Suppose for example it is desired to construct a development Δ with Properties 1, 5 only. The application of operations 2, 3, 4 leads to a class $\mathfrak{P} \equiv [0 \leqq x < \frac{1}{2}$, or $\frac{1}{2} < x \leqq 1$, or $x = 2$, or $x = 3]$, and a development Δ where $\mathfrak{P}^{ml} = [l/2^m \leqq x \leqq (l+1)/m]$ when $l = (1, 2, \cdots, 2^m - 1)$ omitting the values $l = 2^{m-1} - 1$, $l = 2^{m-1}$; $\mathfrak{P}^{ml} = (1/2 - 1/2^m \leqq p < 1/2)$ if $l = 2^{m-1} - 1$, $\mathfrak{P}^{ml} \equiv [1/2 < p \leqq 1/2 + 1/2^m]$ if $l = 2^{m-1}$; $\mathfrak{P}^{ml} \equiv [1/m, 3]$ if $l = 2^m$; and $\mathfrak{P}^{ml} \equiv [0 \leqq x \leqq 1/2^m$ or $x = 2]$ if $l = 0$. That this development does not possess Properties 2, 3, 4 is easily verified.

14. F. Hausdorff‡ has introduced the concept of *topological* space. Such a

* The properties (4), (5), are together equivalent to the statement, limit of a sequence is unique. Cf. § 3, Theorem 2.

† Loc. cit., § 48.

‡ Loc. cit., p. 213.

space consists of a class $\mathfrak{E}$ of points x which correspond to subsets $\mathfrak{U}_x$ of $\mathfrak{E}$ called regions* and are subject to the following conditions:

(A) Each point x implies at least one region $\mathfrak{U}_x$; every region $\mathfrak{U}_x$ contains the point x;

(B) If $\mathfrak{U}_x, V_x$ are two regions belonging to the same point x, then there is a region W_x which is contained in both $\mathfrak{U}_x$ and V_x;

(C) If a point y is contained in $\mathfrak{U}_x$ there is a region $\mathfrak{U}_y$ which is a subset of $\mathfrak{U}_x$;

(D) For every two distinct points x, y there are two regions $\mathfrak{U}_x, \mathfrak{U}_y$ without a common point.

Let $\delta(x, y)$ be a symmetric distance function defined on a class $\mathfrak{E}$. We define regions $\mathfrak{U}_x$ as follows: for every positive number a the class of all points y such that $\delta(x, y) < a$ is a region $\mathfrak{U}_x$ of x. With this definition of region *a necessary and sufficient condition that* $\mathfrak{E}$ *be a topological* space is that:

(1) *If* $\delta(x, y) < a$ *there exist a positive number* b† *such that whenever* $\delta(y, z) < b$ *then* $\delta(x, z) < a$.

(2) *If* x *and* y *are any two points there is a positive number* a *such that* $\delta(x, z) < a$ *implies* $\delta(y, z) > a$.

From the above one easily deduces the conditions under which a development Δ determines a topological space.

The properties (1), (2) of δ stated above do not imply the property of coherence as is shown by the following example. Let $\mathfrak{E}$ denote the points: $0, 1/n\,(n)$ together with points $x = 1+1/n\,(n)$. The function

$$\delta(x, y) = |x - y|,$$

except when $x = 1/n$, $y = 1 + 1/n$, when $\delta(x, y) = 1/n$. The space $\mathfrak{E}$ is compact and topological but not coherent.

The topological space defined by a symmetric distance function δ is not general but satisfies the first enumerability axiom of Hausdorff in the sense that the system $(\mathfrak{U}_x)$ is equivalent to a system (V_x) satisfying this axiom.‡

We shall now state the conditions under which a topological space becomes equivalent to a metric space. In place of axiom (A) we shall employ the stronger axiom:

(A_ω) There exists for each point x a fixed sequence $\{\mathfrak{U}_x^m\}$ of regions of x such that $\mathfrak{U}_x^m$ contains x and $\mathfrak{U}_x^{m+1}$.§

A region $\mathfrak{U}_x$ is of rank m if it is contained in a region $\mathfrak{U}_y^m$ for some point y and is not contained in any region $\mathfrak{U}_y^{m+1}$.

(G) If $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3, \cdots \mathfrak{U}_m \cdots$ is any sequence of regions such that $\mathfrak{U}_m$ is

* Hausdorff uses the term "Umgebung."

† The number b is dependent on x and y.

‡ Cf. Hausdorff, loc. cit., p. 263.

§ This is virtually the assumption of a type of development. The further assumptions make it possible to replace the infinite development by a finite one.

of rank m (at least) and there is a point x common to the classes $\mathfrak{U}_m^0$† then there is for every region $\mathfrak{U}_x$ of x an m_0 such that for every $m \geqq m_0$, $\mathfrak{U}_m$ is contained in $\mathfrak{U}_x$.

(H) If $(\mathfrak{U})$ is any family of domains such that $\mathfrak{E}$ is the aggregate of the classes $\mathfrak{U}$ then there is a finite subfamily $\mathfrak{U}_1, \mathfrak{U}_2, \cdots \mathfrak{U}_n$ of $(\mathfrak{U})$ whose aggregate is $\mathfrak{E}$.

From axioms (A_ω), (G), it follows that *for every region* $\mathfrak{U}_x$ *there is an* m *such that* $\mathfrak{U}_x^m$ *belongs to* $\mathfrak{U}_x$. Every region $\mathfrak{U}_x$ is a domain. It follows readily from axiom (H) that $\mathfrak{E}$, and every closed subset of $\mathfrak{E}$, is compact and admits the Heine-Borel theorem.

From axioms (A_ω) and (H) we have for every positive integer m a finite set of regions;

$$\mathfrak{U}^{m1}, \mathfrak{U}^{m2}, \mathfrak{U}^{m3}, \cdots, \mathfrak{U}^{ml_m};$$

such that

$$\mathfrak{E} = \sum_{l=1}^{l=l_m} \mathfrak{U}^{ml}.$$

The development $\Delta \equiv ((\mathfrak{P}^{ml}))$, *where* $\mathfrak{P}^{ml}$ is the set of α-points of $\mathfrak{U}^{ml}$ is a finite complete development of $\mathfrak{E}$. We shall show that Δ is closed and equivalent to the system $(\mathfrak{U}_x)$ with respect to limit of a sequence. Let $F = \{\mathfrak{P}^{m_n l_n}\}$ be any fundamental sequence of Δ. There is a closed compact class $\mathfrak{F}_n$ common to $\mathfrak{P}^{m_1 l_1}, \cdots \mathfrak{P}^{m_n l_n}$. Since $\mathfrak{F}_n$ contains $\mathfrak{F}_{n+1}$ there must be a point x common to the $\mathfrak{F}_n$. Hence F is closed. Suppose a sequence $\{x_n\}$ is connected with a fundamental sequence F. Without loss of generality we may suppose x_n contained in $\mathfrak{P}^{m_n l_n}$. Then relative to Δ, the sequence has a limit x. Let y be a limit of the sequence in $\mathfrak{E}$. Since every class $\mathfrak{P}^{m_n l_n}$ contains x it follows (Axiom (G)) that for sufficiently large values of m the corresponding regions $\mathfrak{U}^{m_n l_n}$ are contained in a given region $\mathfrak{U}_x$ of x. Since y must be at least an α-point of $\mathfrak{U}^{m_n l_n}$, it follows that y is an α-point of every region $\mathfrak{U}_x$, that is $y = x$, which was to be proved. That any limit in $\mathfrak{E}$ is a limit relative to Δ is evident. Consequently limit is unique relative to Δ and the two systems of limits are the same.

It follows from the conclusions just reached and Theorem 21 that the set $\mathfrak{E}$ is a compact metric space.

THEOREM 31. *A necessary and sufficient condition that a topological space be a compact metric space is that the axioms* (A_ω), (G), (H) *be satisfied.*

The extension of this theory to non-compact spaces will be the topic of a separate discussion.

15. We shall apply the results of the preceding article to spaces $\mathfrak{S}$ satisfying

† $\mathfrak{U}_m^0$ is the class of all the α-points of $\mathfrak{U}_m$. An α-point (Hausdorff) is a point x such that every $\mathfrak{U}_x$ contains a point of $\mathfrak{U}_m$.

the axiom systems Σ_1, Σ_2 of R. L. Moore.* In the systems Σ_1, Σ_2 the undefined elements are points p and classes R of points called regions. A region $\mathfrak{R}$ will be a region $\mathfrak{R}_p$ in the sense of § 14 if $\mathfrak{R}$ contains p. Then $\mathfrak{S}$ is easily shown to be a topological space. We shall establish the further result

THEOREM 32. *A space $\mathfrak{S}$ satisfying axiom systems Σ_1 or Σ_2 is a topological space satisfying the additional axioms* (A_ω), (G). *Any limited closed subset of $\mathfrak{S}$ satisfies axiom* (H).

The axiom systems Σ_1, Σ_2 each consist of eight axioms and differ only in the 6th and 7th. We shall refer to the axioms of Moore as numbered by him. The proof of the above theorem makes use of Axioms 1, 3, 4, 5 only. That Axiom (A_ω) holds is a consequence of Axiom 1 and Theorems 5 and 6 of the article cited. We shall show that Axiom (G) is satisfied. Suppose

$$\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \cdots, \mathfrak{R}_m, \cdots$$

is any set of regions of rank increasing with m (for convenience suppose $\mathfrak{R}_m$ is of rank m), such that a point p is common to the classes R'_m.† We have to show that if $\mathfrak{R}$ is any region containing p there exists an m_0 such that for every $m \geqq m_0$, $\mathfrak{R}_m$ is contained in $\mathfrak{R}$. Since $\mathfrak{R}_m$ is of rank m there is a sequence $\{\mathfrak{K}_{n_m}\}$ of the fundamental regions postulated in Axiom 1 such that $\mathfrak{K}_{n_m}$ contains $\mathfrak{R}_m^{(m)}$ and $L_m n_m = \infty$. Let $\mathfrak{R}_0$ be a region containing p such that for every positive integer m there is an integer $m' > m$ such that $\mathfrak{K}_{n_{m'}}$, is not contained in $\mathfrak{R}_0$. Let q be any element of $\mathfrak{R}_0$ distinct from p (the existence of q is a consequence of Axiom 3 and Theorem 4, loc. cit.). From the second part of Axiom 1 (loc. cit.) there is a positive integer k such that, if n exceeds k and $\mathfrak{K}_n$ contains p, then $\mathfrak{K}'_n$ is a subset of $\mathfrak{R}_0 - q$. Therefore for every m such that $n_m > k$, $\mathfrak{K}'_{n_m}$ is a subset of $\mathfrak{R}_0$. It follows that every region containing p contains $\mathfrak{K}'_{n_m}$ and therefore $\mathfrak{R}'_m$, for sufficiently large values of m. The proposition is proved.

From Theorem 12 (loc. cit.) it follows that Axiom (H) is satisfied by limited closed sets. From Theorem 31 above it follows that every limited closed set is metric. From Axiom 1 (loc. cit.) it follows that $\mathfrak{S}$ is the sum of an enumerable infinity of metric sets.

On the basis of the complete axiom systems Σ_1, Σ_2 we define an infinite development Δ of $\mathfrak{S}$ for which the corresponding distance function is equivalent to an écart on every limited subset of $\mathfrak{S}$, while limit relative to Δ is equivalent to limit relative to the system of regions R.

From Axiom 1 (loc. cit.) there exists an infinite set of regions $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_3$, $\cdots$ $\mathfrak{K}_n$, $\cdots$ such that if p is a point there is a positive integer n such that $\mathfrak{K}_n$

* *On the foundations of plane analysis situs*, these Transactions, vol. XVII (1916), pp. 131–164.

† $\mathfrak{S}$ is perfect, consequently the derived set coincides with the set of α-points.

contains p. Set $\mathfrak{Q}_1 = \mathfrak{K}_1'$. Each point of $\mathfrak{Q}_1$ may be enclosed in a simple closed curve (Theorem 36, loc. cit.). Hence there is a simple closed curve J_1 containing $\mathfrak{Q}_1$ in its interior (Theorem 42, loc. cit.). Denote by $\mathfrak{P}_1$ the set of all points of J_1 and its interior $\mathfrak{J}_1$. Let n_2 be the least index greater than 1 such that $\mathfrak{K}_{n_2}$ is not a subclass of $\mathfrak{P}_1$. Since $\mathfrak{S}$ is connected (Theorem 22, loc. cit.) there is a simple chain connecting $\mathfrak{P}_1$ and $\mathfrak{K}_{n_2}$ of which $\mathfrak{K}_{n_2}$ is the last and possibly only link. Suppose each point of $\mathfrak{P}_1$ and the derived set of this chain enclosed in a simple closed curve. A finite number of such curves is effective. Hence, as before, we have a closed curve J_2 whose interior $\mathfrak{J}_2$ contains $\mathfrak{P}_1$ and $\mathfrak{K}_n$ $(n \leqq n_2)$. Let $\mathfrak{P}_2 = J_2 + \mathfrak{J}_2$. The process leads to the result stated in the following theorem.

THEOREM 33. *The space* $\mathfrak{S} = \sum_{n=1}^{n=\infty} \mathfrak{P}_n$ *where* $\mathfrak{P}_n$ *is a connected closed limited set bounded by a simple closed curve and* $\mathfrak{P}_n$ *is interior to* $\mathfrak{P}_{n+1}$.

In terms of the sequences $\{\mathfrak{K}_n\}$* and $\{\mathfrak{P}_n\}$ we define a development Δ of $\mathfrak{S}$ as follows. Set $n_1 = 1$. Let n_2 be the least value of $n\,(>1)$ such that $\mathfrak{K}_n$ contains a point of $\mathfrak{P}_1$ not in $\mathfrak{K}_1$. Let n_3 be the least value of $n\,(>n_2)$ such that $\mathfrak{K}_n$ contains a point of $\mathfrak{P}_1$ not contained in any region $\mathfrak{K}_{n'}\,(n' \leqq n_2)$. In general, denote by n_{j+1} the least value of $n\,(>n_j)$ such that $\mathfrak{K}_n$ contains a point of $\mathfrak{P}_1$ not in any region of index less than n_{j+1}. Since $\mathfrak{P}_1$ is limited there is a greatest value j_1 of j for which there is an existent class $\mathfrak{K}_{n_j}$ (Heine-Borel theorem). Denote by $\mathfrak{Q}_1$ the class $\sum_{j=1}^{j=j_1} \mathfrak{K}_{n_j}$. $\mathfrak{P}_1$ is interior to $\mathfrak{Q}_1'$. Let n_{j_1+1} be the least value of n such that $\mathfrak{K}_n$ contains a point of $\mathfrak{P}_2$ not in $\mathfrak{Q}_1$ but no point of $\mathfrak{P}_1$, n_{j_1+2} the least value of n such that $\mathfrak{K}_n$ contains a point of $\mathfrak{P}_2$ not in $\mathfrak{K}_{n_{j'}}(j' \leqq j_1+1)$† but no point of $\mathfrak{P}_1$. As before the indices j_1+i have a greatest va ue j_2. There exists, therefore, a sequence of indices n_j and a sequence of indices j_i such that $\mathfrak{P}_i$ is interior to $\mathfrak{Q}_i = \sum_{j=1}^{j=j_i} \mathfrak{K}_{n_j}$ and no region $\mathfrak{K}_{n_j}$ of index $j > j_i$ contains a point of $\mathfrak{P}_i$. Set $\mathfrak{P}^{1j} \equiv \mathfrak{K}_{n_j}'(j = 1, 2, 3, \cdots)$. Let $(\mathfrak{K}_n^1)$ denote the infinite sequence of regions comprising all regions of $(\mathfrak{K}_n)$ which are proper subregions of regions $\mathfrak{K}_{n_j}$. The order of arrangement of the sequence $(\mathfrak{K}_n)$ will be preserved in the sequence $(\mathfrak{K}_n^1)$. The sequence $(\mathfrak{K}_n^1)$ is easily seen to be a fundamental sequence of regions satisfying the conditions of Axiom 1. From the sequences $\{\mathfrak{P}^{1j}\}$, $\{\mathfrak{K}_n^1\}$ we deduce, precisely as before, a sequence $\{\mathfrak{P}^{2j}\}$ and a fundamental set of regions $\{\mathfrak{K}_n^2\}$. The classes $\mathfrak{P}^{2j}$ are closed subclasses of classes $\mathfrak{P}^{1j}$ and at most a finite number of the $\mathfrak{P}^{2j}$ contain points of a given class $\mathfrak{P}^{1j}$. The continuation of the process leads to a system $\Delta \equiv ((\mathfrak{P}^{mj}))$ which is a complete development of S. Since the classes $\mathfrak{P}^{mj}$ are compact and closed, it is easy to see that Δ is closed. Let $\mathfrak{Q}$ be any limited subclass of S. Then $\mathfrak{Q}$ is contained in some

* We shall assume for convenience that no region $\mathfrak{K}_n$ contains a region $\mathfrak{K}_{n'}$, where $n' < n$. This assumption involves no loss in generality under the hypothesis of Axiom 1.

† The existence of the indices n_{j_1+i} follows from Axiom 1.

class $\mathfrak{P}_{\bar{n}}$. At most a finite number ($j_{\bar{n}}$) of the classes $\mathfrak{P}^{1j}$ contain points of any one class $\mathfrak{P}^{1j}$ and therefore at most a finite number of classes $\mathfrak{P}^{2j}$ contain points of $\mathfrak{P}_{\bar{n}}$. Evidently the reduction of Δ relative to $\mathfrak{P}_{\bar{n}}$ (and therefore to $\mathfrak{Q}$) is finite and complete. If $\mathfrak{Q}$ is closed, $\Delta(\mathfrak{Q})$ is also closed (Theorem 6 above). The development Δ is also regular.

We shall prove that limit as defined by Δ is equivalent to limit as defined by region. Let p be a limit of the sequence $\{p_n\}$ in terms of region. Then every region $\mathfrak{R}$ containing p contains all but a finite number of the elements of the sequence $\{p_n\}$. Consequently since there is a fundamental sequence $F \equiv \{\mathfrak{P}^{ml_m}\}$ of Δ such that p is interior to every $\mathfrak{P}^{ml_m}$ it follows that the sequence $\{p_n\}$ is connected with F. Conversely, if a sequence $\{p_n\}$ is connected with a fundamental sequence F_p, we will show that all but a finite number of the elements p_n are contained in any region $\mathfrak{R}$ containing p. Let q be an element of $\mathfrak{R}$ distinct from p. Then from the second part of Axiom 1 there is an n_0 such that for every $n \geqq n_0$, any region $\mathfrak{R}_n$ containing p is, together with its derivative, contained in $\mathfrak{R} - q$. As every class of stage m is a sublcass of stage $m-1$ it follows that if $m > n_0$ then every class of $\{\mathfrak{R}_n^m\}$ carries in $\{\mathfrak{R}_n\}$ an index $n' > n_0$. Hence, for $m > n_0$, every class $\mathfrak{P}^{ml}$ containing p is contained in $\mathfrak{R}$. Since the sequence $\{p_n\}$ is connected with F_p it is at once evident that the sequence has the limit p in terms of region.

It follows from the results just obtained, because of Theorem 21 above, that every closed limited subset of $\mathfrak{S}$ is a compact metric space and that $\mathfrak{S}$ is therefore the sum of an enumerable infinity of compact metric spaces. As the proof that $\mathfrak{S}$ is in fact a metric space does not involve a direct application of developments Δ it is reserved for a separate paper. We summarize the results of this section in the following theorem:

THEOREM 34. *A space $\mathfrak{S}$ satisfying axiom systems Σ_1 or Σ_2 has the form $\mathfrak{S} = \sum_{n=1}^{\infty} \mathfrak{P}_n$ where $\mathfrak{P}_n$ is a connected, closed, limited set bounded by a simple closed curve, and $\mathfrak{P}_n$ is interior to $\mathfrak{P}_{n+1}$. The space* S *admits a regular, complete, closed development Δ whose reduction relative to any limited closed set $\mathfrak{Q}$ is finite, complete, and closed. Furthermore, if $\mathfrak{Q}$ is perfect, $\mathfrak{Q}$ is a set of the type "(E) normale" of Frechet.* Limit of a sequence is the same whether defined in terms of the development Δ or of regions $\mathfrak{R}$.*

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* Cf. Theorem 23. The development Δ is obviously asingular.

ON THE INFLUENCE OF KEYWAYS ON THE STRESS DISTRIBUTION IN CYLINDRICAL SHAFTS*

BY

T. H. GRONWALL

1. When a keyway is cut in a cylindrical shaft, subject to torsion only, and of circular cross section, the maximum stress in the keyed part will considerably exceed that in the full part. To form an estimate of this increase, Filon† has investigated the torsion in a shaft the cross section of which is composed of confocal ellipses and hyperbolas. His results are expressed in infinite series of trigonometric and hyperbolic functions, and their numerical computation is necessarily somewhat laborious. Moreover, his numerical results (l. c., § 22–23) extend only up to a ratio of minor and major axes in the ellipse equal to $\tanh(\pi/2) = 0.917$, and this section differs too much from a circle to allow a sufficiently close numerical estimate of the stress ratio in the latter case.

In the present paper, a cross section will be considered which frequently occurs in practice: the shaft section is taken as a circle, and the keyway is formed by a circular arc intersecting the former orthogonally.

We shall first determine the exact formulas for the stress distribution, and then derive from these an approximate formula, adapted to practical use, in which the ratio of the maximum stresses in the keyed and full parts of the shaft is represented by the expression

$$\frac{2 - \frac{4}{\pi}\frac{b}{a} + \left(2 - \frac{4}{\pi}\right)\frac{b^2}{a^2}}{1 - 4\frac{b^2}{a^2}},$$

where a and b are the radii of shaft and keyway respectively. This expression is exact for $b = 0$ (and thus shows that a flaw or crack in the surface of a circular shaft has the effect of doubling the maximum stress) but is somewhat too great for $0 < b/a \leqq \frac{1}{4}$, this range of the ratio b/a being sufficient for all cases occurring in common practice.

* Presented to the Society, Feb. 26, 1916.

† L. N. G. Filon, *On the resistance to torsion of certain forms of shafting, with special reference to the effect of keyways*, Philosophical Transactions of the Royal Society, London, ser. A, vol. 193 (1900), pp. 309–352.

If a linear formula for the ratio of maximum stresses in the keyed and full parts of the shaft is desired, the above expression may be replaced by

$$2 - \frac{1}{5}\frac{b}{a},$$

which is greater than the preceding one for $0 < a/b \leqq \frac{1}{4}$.

2. With the notations of the figure, it is seen at once that

$$\tan \alpha = \frac{b}{a}, \tag{1}$$

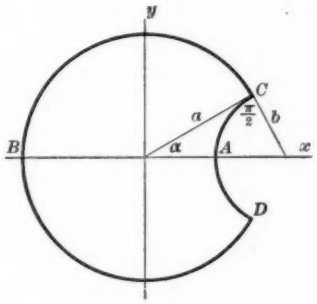

FIG. 1.

and that the distance between centers of shaft and keyway is $a \sec \alpha$.

The first step toward the determination of the torsional stresses consists in the construction of a harmonic function $\Psi(x, y)$* taking the value $\frac{1}{2}(x^2 + y^2)$ on the boundary of our cross section. To this purpose, write $z = x + iy$, $w = u + iv$, and map the area ABCD conformally on the first quadrant in the w-plane in such a manner that C corresponds to $w = 0$, D to $w = \infty$, the circular arc CBD to the u-axis, and the arc CAD to the v-axis. It is seen immediately that

$$w = \frac{z - ae^{\alpha i}}{e^{\alpha i} z - a}, \qquad z = a\frac{w - e^{\alpha i}}{e^{\alpha i} w - 1}, \qquad \frac{dw}{dz} = \frac{1}{a}\frac{(e^{\alpha i} w - 1)^2}{e^{2\alpha i} - 1}, \tag{2}$$

and the segment BA of the real axis corresponds to the first quadrant of the circle $u^2 + v^2 = 1$. From (2) we obtain

$$\tfrac{1}{2}(x^2 + y^2) = \tfrac{1}{2}|z|^2 = \frac{a^2}{2}\frac{1 - 2\cos\alpha\cdot u - 2\sin\alpha\cdot v + u^2 + v^2}{1 - 2\cos\alpha\cdot u + 2\sin\alpha\cdot v + u^2 + v^2}, \tag{3}$$

* The notations are those of Love's *Theory of elasticity*, 2d edition, chapter XIV, except that his ξ, η, X_s, Y_s are replaced by u, v, X, Y.

so that the boundary conditions for Ψ become

$$\Psi = \tfrac{1}{2}|z|^2 = \begin{cases} \tfrac{1}{2}a^2 \text{ for } v = 0 \\ \tfrac{1}{2}a^2\left(1 - \dfrac{4\sin\alpha\cdot v}{1 + 2\sin\alpha\cdot v + v^2}\right) \text{ for } u = 0. \end{cases} \tag{4}$$

Now form Green's function $G(u', v';\ u, v)$ in the first quadrant of the w-plane (u', v' are the coördinates of the variable point and u, v those of the pole). By reflecting the pole successively into the positive v-, the negative u-, and the negative v-axis, we obtain at once

$$G(u', v';\ u, v) = \tfrac{1}{2}\log\frac{[(u'-u)^2 + (v'+v)^2][(u'+u)^2 + (v'-v)^2]}{[(u'-u)^2 + (v'-v)^2][(u'+u)^2 + (v'+v)^2]}.$$

The formula

$$\Psi = \frac{1}{2\pi}\int \Psi_s \frac{\partial G}{\partial n}\, ds$$

now gives

$$\frac{2\pi}{a^2}\Psi = \pi - \int_0^\infty \frac{4\sin\alpha\cdot v'}{1 + 2\sin\alpha\cdot v' + v'^2}\left[\frac{u}{u^2 + (v'-v)^2} - \frac{u}{u^2 + (v'+v)^2}\right] dv'. \tag{5}$$

whence, introducing the function Φ conjugate to Ψ,

$$\begin{aligned} \Phi + i\Psi = {} & \frac{a^2 i}{2} \\ & + \frac{2a^2\tan\alpha}{\pi}\,\frac{\pi - 2\alpha - \pi\cos\alpha\cdot w - (\pi - 2\alpha)\cos 2\alpha\cdot w^2 + \pi\cos\alpha\cdot w^3}{1 - 2\cos 2\alpha\cdot w^2 + w^4} \\ & - \frac{8a^2\sin^2\alpha}{\pi}\,\frac{w^2\log w}{1 - 2\cos 2\alpha\cdot w^2 + w^4} \end{aligned} \tag{6}$$

for $0 < \alpha < \pi/2$, where $\log w$ is the principal value of the logarithm.

For $v = 0$, $w = u$ and we obtain from (6)

$$\begin{aligned} \Phi = {} & \frac{2a^2\tan\alpha}{\pi}\left[\alpha\,\frac{\cos\alpha\cdot u - 1}{1 - 2\cos\alpha\cdot u + u^2} + (\pi - \alpha)\,\frac{\cos\alpha\cdot u + 1}{1 + 2\cos\alpha\cdot u + u^2}\right] \\ & - \frac{8a^2\sin^2\alpha}{\pi}\,\frac{u^2\log u}{1 - 2\cos 2\alpha\cdot u^2 + u^4}, \\ \Psi = {} & \frac{a^2}{2}; \end{aligned} \tag{7}$$

while for $u = 0$, $w = iv$, $\log w = \log v + \pi i/2$ and

$$\begin{aligned} \Phi = {} & \frac{2a^2\tan\alpha}{\pi}\,\frac{(\pi - 2\alpha)(1 + \cos 2\alpha\cdot v^2)}{1 + 2\cos 2\alpha\cdot v^2 + v^4} + \frac{8a^2\sin^2\alpha}{\pi}\,\frac{v^2\log v}{1 + 2\cos 2\alpha\cdot v^2 + v^4}, \\ \Psi = {} & \frac{a^2}{2} - \frac{2a^2\sin\alpha\cdot v}{1 + 2\sin\alpha\cdot v + v^2}; \end{aligned} \tag{8}$$

the identity of these boundary values of Ψ with those given in (4) serves as a check on the calculation.

3. We now proceed to determine the maximum stress in the cross section. The stress components X and Y are given by

$$X = \mu\tau\left(\frac{\partial\Phi}{\partial x} - y\right) = \mu\tau\left(\frac{\partial\Psi}{\partial y} - y\right),$$

$$Y = \mu\tau\left(\frac{\partial\Phi}{\partial y} + x\right) = -\mu\tau\left(\frac{\partial\Psi}{\partial x} - x\right),$$

and the maximum stress is the maximum of $\sqrt{X^2 + Y^2}$, which by an elementary property of harmonic functions must occur on the boundary.* For $u = 0$, we have

$$X = \mu\tau\frac{\partial}{\partial y}[\Psi - \tfrac{1}{2}(x^2 + y^2)]$$

$$= \mu\tau\left\{\frac{\partial}{\partial u}[\Psi - \tfrac{1}{2}(x^2 + y^2)]\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}[\Psi - \tfrac{1}{2}(x^2 + y^2)]\frac{\partial v}{\partial y}\right\}_{u=0}$$

and since $\Psi - \frac{1}{2}(x^2 + y^2) = 0$ for $u = 0$, the partial derivative of this expression in respect to v vanishes, so that

$$X = \mu\tau\frac{\partial u}{\partial y}\frac{\partial}{\partial u}[\Psi - \tfrac{1}{2}(x^2 + y^2)]_{u=0}$$

$$= -\mu\tau\frac{\partial u}{\partial y}\left[\frac{\partial\Phi}{\partial v} + \frac{1}{2}\frac{\partial}{\partial u}(x^2 + y^2)\right]_{u=0}.$$

Similarly

$$Y = -\mu\tau\frac{\partial}{\partial x}[\Psi - \tfrac{1}{2}(x^2 + y^2)]_{u=0}$$

$$= \mu\tau\frac{\partial u}{\partial x}\left[\frac{\partial\Phi}{\partial v} + \frac{1}{2}\frac{\partial}{\partial u}(x^2 + y^2)\right]_{u=0},$$

whence

$$\sqrt{X^2 + Y^2} = \mu\tau\left|\frac{dz}{dw}\right| \cdot \left[\frac{\partial\Phi}{\partial v} + \frac{1}{2}\frac{\partial}{\partial u}(x^2 + y^2)\right]_{u=0}.$$

From (8), (3), and the last of (2), we now obtain

$$\begin{aligned} f(v) &= \frac{\pi\cos\alpha}{\mu\tau a}\sqrt{X^2 + Y^2} \\ &= \frac{2(\pi - 2\alpha)v(\cos 2\alpha + 2v^2 + \cos 2\alpha\cdot v^4)}{(1 + 2\sin\alpha\cdot v + v^2)(1 - 2\sin\alpha\cdot v + v^2)^2} \\ &\quad - \frac{2\sin 2\alpha\,[v(1 + 2\cos 2\alpha\cdot v^2 + v^4) + 2v(1 - v^4)\log v]}{(1 + 2\sin\alpha\cdot v + v^2)(1 - 2\sin\alpha\cdot v + v^2)^2} \\ &\quad + \frac{\pi(1 + \cos 2\alpha)v}{1 + 2\sin\alpha\cdot v + v^2}. \end{aligned} \tag{9}$$

* See Love, *Theory of elasticity*, 2d ed., § 219.

The stress distribution being evidently symmetrical about the x-axis, the stresses on the arc AC are, in particular, equal to those on the arc AD, and this circumstance finds its expression in the fact that (8) is invariant when v is replaced by its reciprocal.

From $f(v) = f(1/v)$ it follows that $f'(v) = -f'(1/v)/v^2$, whence $f'(1) = 0$, so that $f(v)$ becomes a maximum or minimum at $v = 1$. To find any other maxima or minima, it is sufficient to consider the interval $0 \leqq v \leqq 1$; writing

$$v + \frac{1}{v} = 2t, \qquad \frac{dt}{dv} = \frac{1}{2}\left(1 - \frac{1}{v^2}\right) \tag{10}$$

so that $t \geqq 1$, (9) takes the form

$$2f(v) = \frac{Et^2 - 2Ft + G + 2\sin 2\alpha \cdot t\left(v - \frac{1}{v}\right)\log v}{(t + \sin\alpha)(t - \sin\alpha)^2},$$

where

$$E = \pi + (3\pi - 4\alpha)\cos 2\alpha - 2\sin 2\alpha,$$

$$F = \pi \sin\alpha (1 + \cos 2\alpha),$$

$$G = \sin^2\alpha (3\pi - 4\alpha + \pi\cos 2\alpha + 2\sin 2\alpha).$$

The equation $f'(v) = 0$ now becomes

$$\{2(t^2 - \sin^2\alpha)[(E + 2\sin 2\alpha)t - F]$$

$$- (3t + \sin\alpha)[Et^2 - 2Ft + G]\}\left(1 - \frac{1}{v^2}\right)$$

$$+ 2\sin 2\alpha \cdot \log v \left\{(t^2 - \sin^2\alpha)\left[\left(v - \frac{1}{v}\right)\left(1 - \frac{1}{v^2}\right)\right.\right.$$

$$\left.\left. + \left(v + \frac{1}{v}\right)\left(1 + \frac{1}{v^2}\right)\right] - (3t + \sin\alpha)t\left(v - \frac{1}{v}\right)\left(1 - \frac{1}{v^2}\right)\right\} = 0,$$

or reducing by (10) and dividing by $1 - 1/v^2$, thus removing the known root $v = 1$,

$$\begin{aligned}\chi(v) = {} & 2(t^2 - \sin^2\alpha)[(E + 2\sin 2\alpha)t - F] \\ & - (3t + \sin\alpha)[Et^2 - 2Ft + G] \\ & + 8\sin 2\alpha\,[(t^2 - \sin^2\alpha)(2t^2 - 1) \\ & - (3t + \sin\alpha)(t^3 - t)]\frac{\log v}{v - \frac{1}{v}} = 0.\end{aligned} \tag{11}$$

We may prove that this equation has no root between 0 and 1 as follows.

Since any root of $\chi(v) = 0$ in the interval $0 < v < 1$ varies continuously with α, three cases are to be considered when we let α decrease steadily from the given value to zero: (a) there exists some value $\alpha_0 \geqq 0$ of α such that the root in question approaches zero as $\alpha \to \alpha_0$; (b) the root in question approaches the value $v = 1$ for some value of α, or (c) the root will remain within the interval $0 < v < 1$ as α decreases toward zero. It is readily shown that all three cases are impossible. In case (a) we find from (11)

$$\limsup_{\substack{v\to 0\\ \alpha\to\alpha_0}} \frac{\chi(v)}{t^3} = [\,2(E + 2\sin 2\alpha) - 3E\,]_{\alpha=\alpha_0} - 4 \liminf_{\substack{v\to 0\\ \alpha\to\alpha_0}} \sin 2\alpha \cdot \frac{\frac{1}{v}+v}{\frac{1}{v}-v} \log\frac{1}{v},$$

and since

$$\frac{\frac{1}{v}+v}{\frac{1}{v}-v}\log\frac{1}{v} > \log\frac{1}{v} > 1 \quad \text{for} \quad v < e^{-1},$$

we have

$$\liminf_{\substack{v\to 0\\ \alpha\to\alpha_0}} \sin 2\alpha \cdot \frac{\frac{1}{v}+v}{\frac{1}{v}-v}\log\frac{1}{v} \geqq \sin 2\alpha_0,$$

so that finally

$$\limsup_{\substack{v\to 0\\ \alpha\to\alpha_0}} \frac{\chi(v)}{t^3} \leqq -E_{\alpha=\alpha_0} < 0 \quad \text{for} \quad 0 \leqq \alpha_0 < \frac{\pi}{2}.$$

In case (b), we obtain at once

$$\chi(1) = -2\cos^2\alpha\,[\,(2 + 4\sin\alpha)(\pi - \alpha) - 4\sin 2\alpha + \sin^2\alpha(9 - 2\cos\alpha - \pi\sin\alpha - 8\alpha)\,],$$

and since $9 - 2\cos\alpha - \pi\sin\alpha - 8\alpha$ decreases as α increases from 0 to $\pi/2$, we have

$$\chi(1) < -2\cos^2\alpha\,[\,(2 + 4\sin\alpha)(\pi - \alpha) - 4\sin\alpha - (5\pi - 9)\sin^2\alpha\,].$$

In the interval $0 \leqq \alpha \leqq \pi/6$, we find

$$(2 + 4\sin\alpha)(\pi - \alpha) - 4\sin\alpha - (5\pi - 9)\sin^2\alpha > 2(\pi - \pi/6) - 4\cdot\tfrac{1}{2} - (5\pi - 9)\cdot\tfrac{1}{4} > 0,$$

and a similar argument applies to each of the intervals $\pi/6 \leqq \alpha \leqq \pi/3$, $\pi/3 \leqq \alpha \leqq \arcsin 4/5$, $\arcsin 4/5 \leqq \alpha \leqq \pi/2$, so that finally

$$\chi(1) < 0, \qquad 0 \leqq \alpha < \pi/2.$$

Hence $f''(1) < 0$, so that $v = 1$ gives a maximum of $\sqrt{X^2 + Y^2}$. In the third case, it is seen that for $0 < v < 1$

$$\lim_{a \to 0} \chi(v) = -4\pi t^3 < 0,$$

so that none of the three cases yields a root of (11). For $v = 0$, (9) shows that $\sqrt{X^2 + Y^2} = 0$, and since $\chi(v) = 0$ has no root in the interval $0 < v < 1$, the maximum of $\sqrt{X^2 + Y^2}$ occurs at $v = 1$ for $0 \leqq \alpha < \pi/2$, and its value is found from (9) to be

$$\text{(12)} \quad \text{Max. } \sqrt{X^2 + Y^2} = \frac{\mu\tau a}{\pi} \frac{2(\pi - \alpha) - 2\pi \sin\alpha + \pi \sin^2\alpha - 2 \sin\alpha \cos\alpha}{\cos\alpha (1 - \sin\alpha)}.$$

Referring to the figure, the maximum stress on the arc CAD is therefore seen to occur at A, while the stress is zero at C and D. On the remaining part of the boundary, $v = 0$, and here we find, in the same way as for $u = 0$,

$$\sqrt{X^2 + Y^2} = \mu\tau \left| \frac{dw}{dz} \right| \left[\frac{\partial \Phi}{\partial u} + \frac{1}{2} \frac{\partial}{\partial v} (x^2 + y^2) \right]_{v=0}.$$

Treating this expression in exactly the same way as the preceding one, we find a unique maximum value of the stress at $u = 1$ (corresponding to the point B), which however is smaller than (12), the latter therefore giving the absolute maximum.

4. To determine the constant τ, we have the following relation, where M is the moment of the external forces about the axis of the shaft,

$$M = 2\mu\tau \iint [\Psi - \tfrac{1}{2}(x^2 + y^2)]\, dx\, dy.$$

Denoting by $I(u, v)$ the integral in (5), this becomes

$$M = \mu\tau a^2 \iint dx\, dy - \mu\tau \iint (x^2 + y^2)\, dx\, dy - \frac{\mu\tau a^2}{\pi} \iint I(u, v)\, dx\, dy,$$

or changing the integration variables in the last integral and remembering that

$$\frac{\partial(x, y)}{\partial(u, v)} = \left| \frac{dz}{dw} \right|^2$$

which may be found from (2), we obtain

$$\text{(13)} \quad M = \mu\tau a^2 \iint dx\, dy - \mu\tau \iint (x^2 + y^2)\, dx\, dy - \frac{4\mu\tau a^4 \sin^2\alpha}{\pi} \int_0^\infty \int_0^\infty \frac{I(u, v)\, du\, dv}{(1 - 2\cos\alpha \cdot u + 2\sin\alpha \cdot v + u^2 + v^2)^2}.$$

The first integral in (13) equals the area of the cross section, and is most readily found by drawing CD and evaluating the area as the difference of the two circular segments CDBC and CDAC, of radii a and $a \tan \alpha$ and center angles $2\pi - 2\alpha$ and $\pi - 2\alpha$ respectively. Thus we find

$$\iint dx\,dy = (\pi - \alpha + \sin\alpha\cos\alpha)\,a^2 - \left(\frac{\pi}{2} - \alpha - \cos\alpha\sin\alpha\right)a^2\tan^2\alpha$$
$$= a^2\left(\pi - \alpha + \tan\alpha - \frac{\pi}{2}\tan^2\alpha + \alpha\tan^2\alpha\right). \tag{14}$$

The second integral in (13) is the polar moment of inertia; introducing polar coördinates r, θ, the equation of the circular arc of radius b is

$$r = a\sec\alpha\,(\cos\theta - \sqrt{\cos^2\theta - \cos^2\alpha}),$$

and we have

$$\iint (x^2+y^2)\,dx\,dy = 2\int_\alpha^\pi d\theta\int_0^a r^3\,dr + 2\int_0^\alpha d\theta\int_0^r r^3\,dr$$
$$= \tfrac{1}{2}a^4(\pi-\alpha)+\tfrac{1}{2}a^4\sec^4\alpha\int_0^\alpha(\cos\theta - \sqrt{\cos^2\theta-\cos^2\alpha})^4\,d\theta$$
$$= \tfrac{1}{2}a^4(\pi-\alpha) + \tfrac{1}{2}a^4\sec^4\alpha\int_0^\alpha(8\cos^4\theta - 8\cos^2\cos\alpha^2\theta$$
$$+ \cos^4\alpha)\,d\theta + 2a^4\sec^4\alpha\int_0^\alpha\cos\theta\,(2\sin^2\theta - 1$$
$$- \sin^2\alpha)\sqrt{\sin^2\alpha - \sin^2\theta}\,d\theta.$$

Making $\sin\theta = \sin\alpha\sin\phi$ in the last integral, the integrations may be performed, and expressing all trigonometric functions in $\tan\alpha$, we obtain

$$\iint (x^2+y^2)\,dx\,dy = \frac{a^4}{2}\Big[\pi - \alpha + \tan\alpha - \pi\tan^2\alpha + 3\tan^3\alpha$$
$$+ \alpha\tan^2\alpha - \frac{3\pi}{2}\tan^4\alpha + 3\alpha\tan^4\alpha\Big]. \tag{15}$$

In the last integral in (13), $I(u, v)$ may be obtained explicitly from (6), but the further integration in respect to u and v is not possible in finite terms.

5. We shall now derive approximate formulas, adapted to numerical calculation, from the purely theoretical results of the preceding paragraphs, and begin with the expression for the maximum stress. Equation (12) may be written

$$\text{Max. } \sqrt{X^2+Y^2} = 2\mu\tau a - \frac{4\mu\tau a}{\pi}\tan\alpha + \frac{\mu\tau a}{\pi}$$
$$\times\frac{[2\pi - (2\pi-2)\sin\alpha](1-\cos\alpha) - 2\alpha + 2\sin\alpha + (\pi-4)\sin^2\alpha}{\cos\alpha\,(1-\sin\alpha)};$$

replacing α by $\sin\alpha$, the last expression is increased (except for $\alpha = 0$) and we find

$$\text{Max. } \sqrt{X^2+Y^2} \leqq 2\mu\tau a - \frac{4\mu\tau a}{\pi}\tan\alpha + \frac{\mu\tau a}{\pi}\tan^2\alpha\left[\frac{(2\pi-2)\cos\alpha}{1+\cos\alpha} + \frac{2\cos\alpha}{(1-\sin\alpha)(1+\cos\alpha)} + \frac{(\pi-4)\cos\alpha}{1-\sin\alpha}\right].$$

Now each of the three terms in the bracket decreases as α increases (as is seen upon forming their derivatives), and the value of the bracket for $\alpha = 0$ being $2\pi - 4$, we obtain

$$\text{(16)} \qquad \text{Max. } \sqrt{X^2+Y^2} \leqq \mu\tau a\left[2 - \frac{4}{\pi}\tan\alpha + \left(2-\frac{4}{\pi}\right)\tan^2\alpha\right].$$

Next, we proceed to find an upper bound for the third integral in (13). From (5) it follows that, the expression in brackets under the integral sign being evidently positive, we have for $0 < \alpha < \pi/2$

$$I(u,v) < \sin\alpha\int_0^\infty \frac{4v'}{1+v'^2}\left[\frac{u}{u^2+(v'-v)^2} - \frac{u}{u^2+(v'+v)^2}\right]dv',$$

and performing the integration,

$$I(u,v) < \frac{4\pi\sin\alpha\cdot v}{(1+u)^2+v^2}.$$

Hence

$$\int_0^\infty \frac{I(u,v)\,dv}{(1-2\cos\alpha\cdot u+2\sin\alpha\cdot v+u^2+z^2)^2} < \int_0^\infty \frac{4\pi\sin\alpha\cdot v\,dv}{[(1+u)^2+v^2][1-2\cos\alpha\cdot u+u^2+v^2]^2}$$

$$= \frac{\pi\sin\alpha}{1+\cos\alpha}\left[\frac{1}{u}\cdot\frac{1}{1-2\cos\alpha\cdot u+u^2} + \frac{1}{2(1+\cos\alpha)}\cdot\frac{1}{u^2}\log\frac{1-2\cos\alpha\cdot u+u^2}{(1+u)^2}\right]$$

and furthermore

$$\int_0^\infty\int_0^\infty \frac{I(u,v)\,du\,dv}{(1-2\cos\alpha\cdot u+2\sin\alpha\cdot v+u^2+v^2)} < \frac{\pi\sin\alpha}{1+\cos\alpha}\int_0^\infty\left[\frac{1}{u}\cdot\frac{1}{1-2\cos\alpha\cdot u+u^2} + \frac{1}{2(1+\cos\alpha)}\cdot\frac{1}{u^2}\log\frac{1-2\cos\alpha\cdot u+u^2}{(1+u)^2}\right]du.$$

Integrating by parts so as to remove the logarithm, the last integral is readily evaluated, and we finally obtain

$$\int_0^\infty\int_0^\infty \frac{I(u,v)\,du\,dv}{(1-2\cos\alpha\cdot u+2\sin\alpha\cdot v+u^2+v^2)^2} \leqq \frac{\pi(\pi-\alpha-\sin\alpha)(1-\cos\alpha)}{\sin^2\alpha} \tag{17}$$

for $0 \leqq \alpha < \pi/2$, the equality sign holding in (16) and (17) for $\alpha = 0$ only. From (13), (14), (15), and (17) it is now seen that

$$M \geqq \frac{\mu\tau a^4}{2}[\pi-\alpha+\tan\alpha+\alpha\tan^2\alpha-3\tan^3\alpha+3\pi\tan^4\alpha-3\alpha\tan^4\alpha - 8(\pi-\alpha-\sin\alpha)(1-\cos\alpha)].$$

We now use the inequalities

$$\tan\alpha-\tfrac{1}{3}\tan^3\alpha<\alpha<\tan\alpha-\tfrac{1}{3}\tan^3\alpha+\tfrac{1}{5}\tan^5\alpha<\tan\alpha,$$

$$\sin\alpha=\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}>\tan\alpha-\tfrac{1}{2}\tan^3\alpha,$$

$$1-\cos\alpha=1-\frac{1}{\sqrt{1+\tan^2\alpha}}<\tfrac{1}{2}\tan^2\alpha,$$

valid for $0<\tan\alpha<1$, and obtain

$$\begin{aligned} M &\geqq \frac{\mu\tau a^4}{2}[\pi-(\tan\alpha-\tfrac{1}{3}\tan^3\alpha+\tfrac{1}{5}\tan^5\alpha)+\tan\alpha \\ &\quad+(\tan\alpha-\tfrac{1}{3}\tan^3\alpha)\tan^2\alpha-3\tan^3\alpha+3\pi\tan^4\alpha-3\tan\alpha\cdot\tan^4\alpha \\ &\quad-8(\pi-(\tan\alpha-\tfrac{1}{3}\tan^3\alpha)-(\tan\alpha-\tfrac{1}{2}\tan^3\alpha))\cdot\tfrac{1}{2}\tan^2\alpha \\ &=\frac{\mu\pi\tau a^4}{2}\left\{1-\tan^2\alpha\left[4-\frac{19}{3\pi}\tan\alpha\left(1+\frac{9}{19}\tan\alpha-\frac{103}{45}\tan^2\alpha\right)\right]\right\}. \end{aligned}$$

Introducing the assumption $0 \leqq \tan\alpha \leqq \frac{1}{4}$, whence $0 \leqq \alpha \leqq 14^\circ\ 02'\ 10''.48$, the innermost parenthesis is positive, so that finally

$$M \geqq \frac{\mu\pi\tau a^4}{2}(1-4\tan^2\alpha). \tag{18}$$

From (16) and (18) it is seen that, replacing $\tan\alpha$ by b/a, for $0 \leqq b/a \leqq \frac{1}{4}$,

$$\text{Max. } \sqrt{X^2+Y^2} \leqq \frac{2M}{\pi a^3}\cdot\frac{2-\frac{4}{\pi}\frac{b}{a}+\left(2-\frac{4}{\pi}\right)\frac{b^2}{a^2}}{1-4\frac{b^2}{a^2}}.$$

For the full part of the shaft (cross section a circle of radius a) we have the well known formulas

$$\text{Max. } \sqrt{X^2 + Y^2} = \mu\tau a = \frac{2M}{\pi a^3}, \qquad M = \frac{\mu\pi\tau a^4}{2},$$

and consequently *the ratio of maximum stresses in the keyed and full parts of the shaft is less than*

$$\frac{2 - \frac{4}{\pi}\frac{b}{a} + \left(2 - \frac{4}{\pi}\right)\frac{b^2}{a^2}}{1 - 4\frac{b^2}{a^2}}, \tag{19}$$

for $0 < b/a \leqq \frac{1}{4}$, but equal to this expression ($= 2$) for $b = 0$. For practical use, the values of (19) may be tabulated, or we may replace (19) by a linear expression $2 + \lambda(b/a)$, where λ is to be given the least value for which this expression is greater than or equal to (19) throughout the interval $0 \leqq b/a \leqq \frac{1}{4}$. This leads to the condition

$$\lambda \geqq \frac{\left(6 - \frac{4}{\pi}\right)\frac{b}{a} - \frac{4}{\pi}}{1 - 4\frac{b^2}{a^2}}$$

and the right hand member decreasing as b/a increases from 0 to $\frac{1}{4}$, we find

$$\lambda \geqq \frac{\left(6 - \frac{4}{\pi}\right)\cdot\frac{1}{4} - \frac{4}{\pi}}{1 - \frac{1}{4}} = -0.2122,$$

so that the ratio of maximum stresses never exceeds

$$2 - 0.2122\frac{b}{a} \tag{20}$$

in the interval $0 \leqq b/a \leqq \frac{1}{4}$.

SOME CONVERGENT DEVELOPMENTS ASSOCIATED WITH IRREGULAR BOUNDARY CONDITIONS*

BY

JAMES W. HOPKINS

INTRODUCTORY NOTE. The author whose name appears at the head of this paper was a graduate student at Harvard when the United States entered the war. He went to an officers' training camp at the earliest opportunity, received a commission, and has been in France since the fall of 1917. Although his mathematical work was thereby interrupted before he was ready to regard even a first undertaking as definitely accomplished, some of the results that he had obtained are so striking, and, in their field, important, that I have asked and obtained his permission to publish them without further delay. Even of the work that he had done, only a part is contained in the manuscript that I now have, the rest having been lost in the confusion of his departure. The part that is preserved, forming the substance of the present paper, deals with a particular differential system, the place of which in a more general theory is indicated below. Mr. Hopkins had treated other particular systems in a similar way, and, when the work was suspended, was trying to raise the discussion a step higher in the scale of generality; he met with considerable difficulty in this attempt, and it is to be hoped that he will have an opportunity of coming back to the problem.

As the manuscript that I have is only a first rough draft, written before the declaration of war, with no thought of immediate publication, it would have required a considerable amount of editing, in any event. In view of this, I have taken the liberty of rewriting it altogether, in quite a different form,† which seems to me more elegant for the problem in hand, though less adapted to the sort of generalization that Mr. Hopkins had ultimately in view. The exposition, then, is the work of the undersigned, who must in particular assume responsibility for any imperfections that may be found in it; the substance, however, is essentially the work of Mr. Hopkins, and is rightfully published under his name.

DUNHAM JACKSON.

August 1, 1918.

1. PLACE OF THE PROBLEM IN A GENERAL THEORY

Birkhoff‡ has discussed in general terms the development of an arbitrary function $f(x)$ in a series of characteristic functions obtained from an ordinary linear differential equation of the nth order. He proves that if the boundary conditions associated with the differential equation are of the sort that he calls "regular," the expansion will converge, provided that $f(x)$ satisfies

* Presented to the Society, April 27, 1918.

† Mr. Hopkins adheres more closely to Birkhoff's method of treatment in the convergence proof, using the Green's function in determinant form throughout. The desirability of a more compact notation for the treatment of the present problem was urged by Professor Bôcher.

‡ These Transactions, vol. 9 (1908), pp. 373–395.

hypotheses of the order of those usually imposed for the sake of insuring convergence of a Fourier's series. It has been shown* that the distinction between regular and irregular boundary conditions is an essential one, and not merely an accident of Birkhoff's treatment. In a large class of problems with irregular conditions, the expansion of a function having merely a limited number of continuous derivatives can not possibly converge in the usual way,† and the expansion even of an analytic function is likely to be rapidly divergent. In order to lead to a uniformly convergent series, the function $f(x)$ must be of decidedly more special type. The following discussion reveals the exact nature of the specialization, in the case of the series given by a particular differential equation of the third order, with a particular set of irregular boundary conditions. While the scope of the paper is therefore decidedly limited, in comparison with the extent of the field that opens itself for investigation, it gives for the first time a constructive account of a type of series which have some of the properties of power series on the one hand and of trigonometric series on the other, and which possess further novel and striking properties of their own.

2. Necessary condition for convergence

The starting point of the discussion is the differential equation‡

$$\frac{d^3u}{dx^3} + \rho^3 u = 0,$$

with the boundary conditions

$$u(0) = 0, \quad u'(0) = 0, \quad u(\pi) = 0.$$

Its characteristic numbers are real, and are distributed in accordance with the formula

$$\rho_n = C + \frac{2n}{\sqrt{3}} + \epsilon_n,$$

where C is independent of n, and $\lim_{n=\infty} \epsilon_n = 0$. With the notation

$$\delta(t) = e^{-t} + \omega e^{-\omega t} + \omega^2 e^{-\omega^2 t}, \tag{1}$$

where ω is one of the complex cube roots of unity,

$$\omega = e^{2\pi i/3},$$

* D. Jackson, Proceedings of the American Academy, vol. 51 (1915-16), pp. 383–417.

† That is, can not converge uniformly to the value of the function.

‡ For the introductory statements about this differential system, cf. Jackson, loc. cit., pp. 384–393.

the nth characteristic function is

$$u_n(x) = \delta(\rho_n x). \tag{2}$$

The problem is to determine under what circumstances the formal development of an arbitrary function $f(x)$ in a series of the form

$$\sum_{n=1}^{\infty} a_n u_n(x) \tag{3}$$

will converge uniformly to the value of the function. We shall obtain a necessary condition on the function $f(x)$, and then shall show that, in their principal features at least, hypothesis and conclusion can be interchanged. It is admitted that a clean-cut necessary and sufficient condition is beyond the limits of this paper; but an essential part of such a condition, to all appearances the most significant part, is clearly brought to light.

In establishing the necessity of the condition, the differential system which led to the series, and the corresponding theory, for which reference has been made to another paper, may be left altogether in the background. For following all the work that is done here, it is enough to take the series (3) on its own merits, with the explicit formula for $u_n(x)$, and the observation that ρ_n is a positive real number, of the order of magnitude of n.

Suppose that the series (3) converges uniformly throughout an interval $x_1 \leqq x \leqq x_0$, however small, where $0 \leqq x_1 < x_0$. We shall be concerned primarily with the behavior of the series in the interval $(0, \pi)$, but for the purposes of the present section, x_0 may be less than, equal to, or greater than π. There will be a constant g such that

$$|a_n u_n(x)| \leqq g \tag{4}$$

throughout the interval, for all values of n. Although the expression for $u_n(x)$ given above involves imaginaries, the function itself is real, and may be represented by the equivalent formula

$$u_n(x) = e^{-\rho_n x} + 2e^{\frac{1}{2}\rho_n x} \cos\left(\frac{\sqrt{3}}{2}\rho_n x - \frac{2\pi}{3}\right).$$

The first term on the right is exceedingly small, for large values of n; the order of magnitude of the whole expression is that of $e^{\rho_n x/2}$. More precisely, $u_n(x)$ has this order of magnitude, unless the trigonometric factor happens to be zero or nearly zero. But if n is large, the oscillations of the cosine will be so rapid that there will surely be at least one point x_n' between x_1 and x_0 at which it is equal to $+1$, and at such a point (even the small term being positive)

$$u_n(x_n') > 2e^{\frac{1}{2}\rho_n x_n'} > 2e^{\frac{1}{2}\rho_n x_1}.$$

Since (4) holds for $x = x_n'$, it may be inferred, for each value of n from a certain point on, that

$$|a_n| < \tfrac{1}{2}ge^{-\frac{1}{2}\rho_n x_1}. \tag{5}$$

The number x_1 was supposed given. But if the series converges uniformly for $x_1 \leqq x \leqq x_0$, it converges uniformly for $\xi_1 \leqq x \leqq x_0$, if ξ_1 is any number between x_1 and x_0; and the reasoning may be applied to the interval (ξ_1, x_0). So the inequality (5) holds, if x_1 is replaced by ξ_1, except that more terms may have to be left out of account at the beginning. Even these terms, finite in number, may be brought under the formula, if a suitably chosen larger constant, presumably dependent on ξ_1, is substituted for $g/2$. With a slight change in notation—since we have been saying, practically, that x_1 might have had any larger value, below x_0, in the first place, and the inequality is merely strengthened if x_1 is replaced by a smaller value—we may assert that

If the series (3) *converges uniformly throughout any interval reaching out to the point* x_0, *and if* x_1 *is any number less than* x_0, *there will be a number* h, *independent of* n, *such that*

$$|a_n| < he^{-\frac{1}{2}\rho_n x_1}$$

for all values of n.

It will be seen that such a restriction on the coefficients implies convergence of the series for an extended range of complex as well as of real values of x.

Let x_2 be any positive number less than x_1. Setting $x = \xi + \eta i$, consider the magnitude of the three terms which make up $u_n(x)$, according to (1) and (2). We shall have

$$|e^{-\rho_n x}| = e^{-\rho_n \xi} \leqq e^{\frac{1}{2}\rho_n x_2} \tag{6}$$

if $\xi \geqq -\frac{1}{2}x_2$, that is, if x is any point on or to the right of the vertical line $\xi = -\frac{1}{2}x_2$. Similarly,

$$|e^{-\omega\rho_n x}| \leqq e^{\frac{1}{2}\rho_n x_2}, \tag{7}$$

provided that ωx satisfies the same condition, that is, provided that x is on or below the line obtained by rotating the vertical line about the origin as a center through 120° in the negative direction. In the same way,

$$|e^{-\omega^2\rho_n x}| \leqq e^{\frac{1}{2}\rho_n x_2}, \tag{8}$$

if x is on or above the line into which the vertical line is carried by a rotation of 120° in the positive direction.* These three lines bound an equilateral triangle, with its center at the origin, and with one vertex at the point x_2. If x is within or on the boundary of the triangle, the relations (6), (7), and (8)

* More formally,

$$|e^{-\rho_n x}| = e^{-\rho_n \xi}, \qquad |e^{-\omega\rho_n x}| = e^{\frac{1}{2}\rho_n(\xi + \sqrt{3}\eta)}, \qquad \text{and} \qquad |e^{-\omega^2\rho_n x}| = |e^{\frac{1}{2}\rho_n(\xi - \sqrt{3}\eta)};$$

and these three quantities will be less than or equal to $e^{\rho_n x_2/2}$, if $\xi \geqq -\frac{1}{2}x_2$, $\xi + \sqrt{3}\eta \leqq x_2$, $\xi - \sqrt{3}\eta \leqq x_2$, respectively.

all subsist at once;

$$|u_n(x)| \leqq 3e^{\frac{1}{2}\rho_n x_2},$$

and

$$|a_n u_n(x)| \leqq 3he^{\frac{1}{2}\rho_n(x_2 - x_1)}.$$

Since $x_2 - x_1$ is negative, and since ρ_n is of the order of magnitude of n, the right-hand member of the last inequality is the general term of a convergent series of positive terms. This means that the series (3) converges uniformly throughout the triangle. It will be recalled that x_2 is any number less than x_1, and x_1 any number less than x_0. Therefore we may say that

The series represents a function which is analytic throughout the interior of an equilateral triangle having its center at the origin, and one vertex at the point x_0.

It is to be noticed that if X is the greatest possible value of x_0, or the upper limit of possible values of x_0, the corresponding triangle is the actual region of convergence of the series, except possibly for points on the positive axis of reals, and on the rays inclined at 120° and 240°, beyond the range of uniform convergence. That is, the series can not converge at any point outside the triangle and not lying on one of these three rays. Suppose it were convergent at such a point, x_2'. For $x = x_2'$, one of the three terms of $u_n(x)$ will be so much larger than either of the others as to determine the order of magnitude of the whole expression; there is no possibility of two terms destroying each other. Let the triangle determined by X be magnified until one side passes through x', and let X' be the right-hand vertex of the new triangle. The term of $u_n(x)$ which is largest at x_2' will have the same absolute value all along the side through x_2', and along each of the other two sides the term which outweighs the others will have this same absolute value still. Since the absolute value of the preponderant term is essentially the absolute value of $u_n(x)$, this means that $u_n(x)$ has the same order of magnitude all around the perimeter of the triangle, except that it may be *smaller* at a vertex, because of the possibility of cancellation there. In particular, $u_n(X')$ does not exceed $u_n(x_2')$ in order of magnitude. Since $a_n u_n(x_2')$ remains finite, the same is true of $a_n u_n(X')$. Therefore, by reasoning similar to that already employed, the series converges uniformly for $0 \leqq x \leqq x_0'$, if x_0' is any number less than X'; and this is inconsistent with the hypotheses, since x_0' might be taken greater than X.

Let the sum of the series be denoted by $f(x)$. Further properties of this function, beyond its mere analytic character, can be deduced from the special form of the individual terms. If any one of the functions $u_n(x)$ is expanded in a power series, by combining the series for the three exponentials of which it is made up, two thirds of the terms will cancel, because $1 + \omega + \omega^2 = 0$, and only the second, fifth, eighth, . . ., powers of x will remain. So it appears, formally at least, that $f(x)$ will be similarly constituted. This may be verified as follows.

In the first place, $f(0) = f'(0) = 0$, because $u_n(0) = u_n'(0) = 0$, for every value of n, and the series can be differentiated term by term. It follows that we can write

$$f(x) = x^2 \phi(x),$$

where $\phi(x)$ is analytic in the same region as $f(x)$. By direct substitution it it seen that $u_n(\omega x) = \omega^2 u_n(x)$, for all values of n, whence $f(\omega x) = \omega^2 f(x)$. For ϕ this means that

$$\phi(\omega x) = f(\omega x)/(\omega^2 x^2) = f(x)/x^2 = \phi(x),$$

and so it may be inferred* that $\phi(x)$ can be represented by a power series in x^3.

Thus the necessary condition with regard to the form of $f(x)$ is established. We may express it by saying that

$$f(x) = x^2 \psi(x^3),$$

where ψ is a single-valued analytic function of its argument, or by saying that $f(x)$ is analytic, $f(0) = f'(0) = 0$, and $f''(x)$ is a single-valued function of x^3. The necessity of the condition, in an appropriate region, has been proved for uniform convergence over any interval of the positive axis of reals. To separate the results most closely related to the remainder of the discussion from those to which we shall not have occasion to refer again, we may summarize in two stages, thus

THEOREM I. *If the series* (3) *converges uniformly throughout the interval* $0 \leqq x \leqq x_0$, *where* $0 < x_0 \leqq \pi$, *it represents a function* $f(x)$ *which is analytic throughout the interior of an equilateral triangle having its center at the origin and one vertex at the point* $x = x_0$, *and which involves in its power series expansion only powers of index congruent to* 2, *mod.* 3.

THEOREM Ia. *In the hypothesis of Theorem I, the interval* $(0, x_0)$ *may be replaced by any interval* (x_1, x_0), *where* $0 < x_1 < x_0$, *and the restriction* $x_0 \leqq \pi$ *is unnecessary.*

3. PROOF OF CONVERGENCE FOR THE FUNCTION x^2

The simplest function satisfying the necessary conditions of Theorem I is the function x^2. The purpose of this section is to prove

THEOREM II. *The formal development of* x^2 *in a series of the form* (3) *converges uniformly to the value* x^2 *for*† $0 \leqq x \leqq x_0$, *if* $0 < x_0 < \pi$.

* Let $\varphi(x)$ be expanded according to powers of x; since $\varphi(\omega^2 x) = \varphi(\omega x) = \varphi(x)$, we may write $\varphi(x) = \frac{1}{3}[\varphi(x) + \varphi(\omega x) + \varphi(\omega^2 x)]$, and if we substitute the series for φ inside the bracket, all the powers of x except the third, sixth, ninth, $\cdots$, will cancel.

† The series of course converges to the value 0 for $x = \pi$, since each term vanishes separately; and from this it follows that the convergence can not be uniform for the whole interval $(0, \pi)$.

We must refer to the general theory of boundary value and expansion problems once more for the formulas on which the proof is based. But here again, as in the preceding section, we can write down the necessary formulas at the start, and then use them on their own merits, without further reference to their derivation.

The coefficients in the formal expansion of an arbitrary function $f(x)$ in a series (3) are given by integrals analogous to those for the coefficients in a Fourier's series; it is not necessary even to write them down here. On the other hand, the following expression for the sum of the first n terms of the series is essential*:

$$I_n(x) = \frac{1}{2\pi i}\int_{\gamma_n} d\rho \int_0^\pi 3\rho^2 G(x, s, \rho) f(s)\, ds. \tag{9}$$

Here γ_n indicates a path in the ρ-plane, consisting of an arc of 120° of a circle about the origin, of such radius as to include just the first n of the characteristic numbers ρ_i. It may be thought of as extending in the positive direction from $-R_n\omega$ to $-R_n\omega^2$, say, if R_n is the radius of the circle. It corresponds to a complete circuit in the plane of the variable ρ^3, which is the parameter that actually appears in the differential equation. The function G is the Green's function of the differential system. Its form may be found by evaluating the general determinant expression † for such a function, as applied to the problem in hand. The reduction to a compact formula involves a certain amount of computation, but it is fairly straightforward, and there is perhaps no need of giving the details of the process.‡ The result is conveniently expressed in terms of the function δ, introduced in (1). It has different forms, according as x is greater or less than s:

$$\begin{aligned} 3\rho^2 G &= \frac{\delta(\rho\overline{x-s})\,\delta(\rho\pi) - \delta(\rho\overline{\pi-s})\,\delta(\rho x)}{\delta(\rho\pi)} && \text{if} \quad x > s, \\ 3\rho^2 G &= -\frac{\delta(\rho\overline{\pi-s})\,\delta(\rho x)}{\delta(\rho\pi)} && \text{if} \quad x < s. \end{aligned} \tag{10}$$

* Birkhoff, loc. cit., pp. 379, 390.

† Birkhoff, loc. cit., p. 378.

‡ Some of the principal steps are these: Using the notation and formulas of Birkhoff, loc. cit., pp. 378, 391, let $y_1(x) = e^{-\rho x}$, $y_2(x) = e^{-\omega\rho x}$, $y_3(x) = e^{-\omega^2\rho x}$. It is found that $\bar{y}_1(s) = e^{\rho s}/(3\rho^2)$, $\bar{y}_2(s) = \omega e^{\omega\rho s}/(3\rho^2)$, $\bar{y}_3(s) = \omega^2 e^{\omega^2\rho s}/(3\rho^2)$, so that

$$\bar{G} = \pm\,\delta(\rho\overline{x-s})/(6\rho^2),$$

the sign being $+$ for $x > s$, $-$ for $x < s$. The values of the functions $\bar{y}_i$ are most easily verified by noticing that they satisfy the equations $\Sigma y_i\,\bar{y}_i = 0$, $\Sigma y_i'\,\bar{y}_i = 0$, $\Sigma y_i''\,\bar{y}_i = 1$, the index i running from 1 to 3. The value of $\bar{G}$ is substituted in the formula for G. In the determinant which forms the numerator, suitable multiples of the first three columns are added to the last column, so as to reduce the second and third of its elements to zero, and, if $x < s$, the first element as well. It is easy then to evaluate the determinant and finish the computation.

It will be seen that these expressions differ only by an additive term $\delta(\overline{\rho x - s})$; but it is convenient to write them, as we have done, with a common denominator throughout.

The problem is, *with the formulas* (10) *for the Green's function, and with the substitution of* s^2 *for* $f(s)$ *in* (9), *to show that*

$$\lim_{n=\infty} I_n(x) = x^2,$$

uniformly in $(0, x_0)$.

The integration with regard to s in (9) can be performed explicitly. In (10), only the quantities $\delta(\overline{\rho x - s})$ and $\delta(\overline{\rho\pi - s})$ depend on s; the latter appears in the integrand throughout the whole interval from 0 to π, the former only while s goes from 0 to x. By integration by parts,

$$\int_0^x s^2 e^{\rho(s-x)}\,ds = \frac{x^2}{\rho} - \frac{2x}{\rho^2} + \frac{2}{\rho^3} - \frac{2}{\rho^3}e^{-\rho x},$$

$$\int_0^x s^2 e^{\omega\rho(s-x)}\,ds = \frac{x^2}{\omega\rho} - \frac{2x}{\omega^2\rho^2} + \frac{2}{\rho^3} - \frac{2}{\rho^3}e^{-\omega\rho x},$$

$$\int_0^x s^2 e^{\omega^2\rho(s-x)}\,ds = \frac{x^2}{\omega^2\rho} - \frac{2x}{\omega\rho^2} + \frac{2}{\rho^3} - \frac{2}{\rho^3}e^{-\omega^2\rho x},$$

so that, with the relation $1 + \omega + \omega^2 = 0$ again,

$$\int_0^x s^2\,\delta(\overline{\rho x - s})\,ds = \frac{3x^2}{\rho} - \frac{2}{\rho^3}\delta(\rho x).$$

As far as this computation is concerned, x might have any value whatever; in particular,

$$\int_0^\pi s^2\,\delta(\overline{\rho\pi - s})\,ds = \frac{3\pi^2}{\rho} - \frac{2}{\rho^3}\delta(\rho\pi).$$

It is at this point that the very special form of the function to be developed has its effect. It we had performed the same calculation with 1 or s in place of s^2, we should not have found precisely $\delta(\rho x)$ and $\delta(\rho\pi)$ for the concluding terms on the right, but other combinations of the exponentials instead, and the cancellation of these terms at the next stage, essential for the success of the convergence proof, would not take place. As it is, we see that

$$\delta(\rho\pi)\int_0^x s^2\,\delta(\overline{\rho x - s})\,ds - \delta(\rho x)\int_0^\pi s^2\,\delta(\overline{\rho\pi - s})\,ds = \frac{3x^2}{\rho}\delta(\rho\pi) - \frac{3\pi^2}{\rho}\delta(\rho x), \tag{11}$$

that is,

$$\int_0^\pi 3\rho^2 Gs^2\,ds = \frac{3x^2}{\rho} - \frac{3\pi^2}{\rho}\cdot\frac{\delta(\rho x)}{\delta(\rho\pi)}, \tag{12}$$

and the most formidable terms have disappeared. It is to be remembered that the computation so far is exact; we have not dismissed as negligible anything that is actually present. We are ready now, however, to examine carefully the order of magnitude of the quantities that we are dealing with.

Let $\delta(\rho\pi)$ be written in the form

$$\delta(\rho\pi) = e^{-\rho\pi} + \omega e^{-\omega\rho\pi}(1 + \omega e^{\sqrt{3}\rho\pi i}).$$

The parenthesis is a periodic function of ρ, having the real period $2/\sqrt{3}$. It becomes infinite uniformly as the imaginary part of ρ becomes negatively infinite, and uniformly approaches 1 when the imaginary part of ρ becomes positively infinite. It vanishes at the points

$$\rho_n' = \frac{1}{3\sqrt{3}} + \frac{2n}{\sqrt{3}}, \qquad n = 0, \pm 1, \pm 2, \cdots,$$

and nowhere else in the ρ-plane. Consequently, if small circles of constant radius ϵ are described about the points ρ_n', and the interiors of these circles are removed from the ρ-plane, the absolute value of the parenthesis will have a positive minimum k_1 in the part of the plane that is left.

Let S' denote the sector of the ρ-plane for which $-\pi/3 \leqq \arg\rho \leqq 0$, and S'' the sector for which $0 \leqq \arg\rho \leqq \pi/3$. Each of the arcs γ_n crosses both of these sectors; let its halves be denoted by γ_n' and γ_n'' respectively. For large values of n, the characteristic numbers ρ_n will be inside the little circles of radius ϵ mentioned in the preceding paragraph,* and the arcs may be supposed drawn so as not to cut any of these circles. On γ_n'', $e^{-\rho\pi}$ approaches zero uniformly, and $e^{-\omega\rho\pi}$ becomes uniformly infinite, as n increases without limit. So, if k is any positive constant less than k_1,

$$|\delta(\rho\pi)| > k|e^{-\omega\rho\pi}|$$

on this arc, for all values of n from a certain point on.

On the other hand,

$$|\delta(\rho x)| \leqq 3|e^{-\omega\rho x}| \leqq 3|e^{-\omega\rho x_0}|$$

for $0 \leqq x \leqq x_0$ and for ρ in S''. On γ_n'', therefore,

$$\left|\frac{\delta(\rho x)}{\delta(\rho\pi)}\right| < \frac{3}{k}|e^{\omega\rho(\pi - x_0)}|,$$

which approaches zero uniformly along the arc. Consequently

$$\lim_{n=\infty} \int_{\gamma_n''} \frac{\delta(\rho x)}{\rho\delta(\rho\pi)}\, d\rho = 0,$$

* It is not asserted that ρ_n will be in the circle about the point ρ_n' with the same subscript n. So far as the writer knows, there may be a constant difference between these subscripts, but that is inessential for the main argument.

uniformly for $0 \leqq x \leqq x_0$; the factor ρ in the denominator would compensate for the increase in the length of the arc with n, even if the convergence of the exponential were not already strong enough without it.

The conclusion can be extended at once to the integral over γ_n', since the integrand takes on conjugate imaginary values for conjugate imaginary values of ρ.

The proof of the theorem is now practically immediate. For the ρ-integral of the first term on the right in (12) is independent of n:

$$\int_{\gamma_n} \frac{3x^2}{\rho} d\rho = 3x^2 \int_{\gamma_n} \frac{d\rho}{\rho} = 3x^2 [\log \rho]_{-K_n\omega^2}^{K_n\omega^2} = 3x^2 \cdot \frac{2\pi i}{3};$$

so that, uniformly for $0 \leqq x \leqq x_0$,

$$\lim_{n=\infty} \int_{\gamma_n} d\rho \int_0^\pi 3\rho^2 Gs^2 \, ds = 2\pi i x^2,$$

which is equivalent to the relation to be proved.

It is scarcely more difficult to prove the convergence for x^5 or x^8, or any other single power of the form x^{3m+2}. For example,

$$\int_0^x s^5 \delta(\rho\overline{x-s}) \, ds = \frac{3x^5}{\rho} - \frac{3 \cdot 5 \cdot 4 \cdot 3x^2}{\rho^4} + \frac{5!}{\rho^6} \delta(\rho x),$$

$$\int_0^\pi 3\rho^2 Gs^5 \, ds = \frac{3x^5}{\rho} - \frac{3\pi^5}{\rho} \frac{\delta(\rho x)}{\delta(\rho\pi)} - \frac{3 \cdot 5 \cdot 4 \cdot 3x^2}{\rho^4} + \frac{3 \cdot 5 \cdot 4 \cdot 3\pi^2}{\rho^4} \frac{\delta(\rho x)}{\delta(\rho\pi)}. \tag{13}$$

When integrated with regard to ρ, the first term on the right in (13) gives $2\pi i x^5$. The second and fourth terms give results that approach zero, because of the relative magnitudes of $\delta(\rho x)$ and $\delta(\rho\pi)$. In the case of the third term, the magnitude of the denominator would be enough to bring about convergence, even without the fact that

$$\int_{\gamma_n} \frac{d\rho}{\rho^4} = 0$$

exactly. So there is no new difficulty here, except the greater length of the formulas.

The methods of the present section, then, are sufficient to establish the convergence of the formal development for any polynomial involving only the powers of x designated in Theorem I. For a function represented by an infinite series of these powers, some new considerations are needed, which it is the purpose of the following section to supply.

4. Proof of convergence for the general function $x^2\psi(x^3)$

The final theorem of the paper is the following.

Theorem III. *If $f(x)$ is any function which is analytic throughout the interior and on the boundary of a circle of radius x_0 about the origin, where $0 < x_0 < \pi$, and which involves in its power series expansion only powers of index congruent to 2, mod. 3, and which has furthermore a continuous second derivative for real values of x throughout the whole interval $0 \leqq x \leqq \pi$, then the formal development of $f(x)$ converges uniformly to the value of the function for $0 \leqq x \leqq x_0$.*

In various respects this falls short of being a complete converse of Theorem I. It restricts x_0 to values less than π, necessarily, as was seen in connection with the function x^2; but more than this, it presupposes the analytic character of $f(x)$, not merely throughout the triangle of Theorem I, but throughout the circle circumscribed about this triangle; and it includes an assumption relative to the interval (x_0, π). It would not be hard to do something toward reducing these discrepancies, but in the present paper we shall be satisfied with the theorem as stated.* Clearly some assumption must be made covering the whole interval $(0, \pi)$, in order that the integrals involved may have a meaning.

We use the general formulas of the preceding section, in particular the expression (9) for the sum of n terms of the series, and follow the same method of procedure as far as may be.

It is no longer possible to carry through the s-integration to a finish. Integrating by parts twice, and remembering that $f(0) = f'(0) = 0$, we find that

$$\int_0^x f(s)\, e^{\rho(s-x)}\, ds = \frac{1}{\rho} f(x) - \frac{1}{\rho^2} f'(x) + \frac{1}{\rho^2}\int_0^x f''(s)\, e^{\rho(s-x)}\, ds,$$

with corresponding formulas for the integrals involving $e^{\omega\rho(s-x)}$ and $e^{\omega^2\rho(s-x)}$. By combination of the three equations,

$$\int_0^x f(s)\, \delta(\rho\overline{x-s})\, ds = \frac{3f(x)}{\rho} + \frac{1}{\rho^2}\int_0^x f''(s)\,[e^{\rho(s-x)} + \omega^2 e^{\omega\rho(s-x)} + \omega e^{\omega^2\rho(s-x)}]\, ds.$$

The bracket on the right differs from a δ-function by the interchange of the coefficients ω and ω^2 before the exponentials. Let it be denoted by $\delta_1(\rho\overline{x-s})$. Since the computation is valid† for $x = \pi$,

* With regard to these secondary hypotheses, Mr. Hopkins's own statement is in some respects less general and in some respects more general than that in the text. The writer must admit that he has not studied the proof by the other method carefully enough to be sure how far the differences are significant for the respective methods, and how far they are accidental.

† It is for the integration by parts at this stage that the second derivative of f is used all the way from 0 to π.

$$\delta(\rho\pi)\int_0^x f(s)\,\delta(\rho\overline{x-s})\,ds - \delta(\rho x)\int_0^\pi f(s)\,\delta(\rho\overline{\pi-s})\,ds$$

$$(14)\qquad = \frac{3f(x)}{\rho}\delta(\rho\pi) - \frac{3f(\pi)}{\rho}\delta(\rho x)$$

$$+\frac{1}{\rho^2}\left[\delta(\rho\pi)\int_0^x f''(s)\,\delta_1(\rho\overline{x-s})\,ds - \delta(\rho x)\int_0^\pi f''(s)\,\delta_1(\rho\overline{\pi-s})\,ds\right].$$

This corresponds to the formula (11) in the preceding section, except that for $f(s) = s^2$ the bracket reduces to zero. It will be found possible in the present case, by making use of the special properties of the function f, to break up the bracket into a number of terms, so that two of them will cancel, while the others are of such order of magnitude as not to interfere with the convergence.

Before undertaking this, however, let us dispose of the simpler terms in (14). Taking account of the denominator in (10), and denoting the bracket in (14) by A, we have

$$\int_0^\pi 3\rho^2 Gf(s)\,ds = \frac{3f(x)}{\rho} - \frac{3f(\pi)}{\rho}\cdot\frac{\delta(\rho x)}{\delta(\rho\pi)} + \frac{A}{\rho^2\,\delta(\rho\pi)}.$$

All this has to be integrated with regard to ρ over the arc γ_n, to give $2\pi I_n(x)$. The result of integrating $3f(x)/\rho$ is $2\pi i f(x)$ exactly. The ρ-integral of $3f(\pi)\,\delta(\rho x)/[\rho\delta(\rho\pi)]$ approaches zero, uniformly for $0 \leqq x \leqq x_0$, when n becomes infinite, by the reasoning of the preceding section. It remains to show that the integral of $A/[\rho^2\,\delta(\rho\pi)]$ also converges to zero.

Let the first integral in the bracket A be written as a sum of three, involving respectively the three terms of $\delta_1(\rho\overline{x-s})$. Consider the second of these integrals

$$\int_0^x f''(s)\cdot\omega^2 e^{\omega\rho(s-x)}\,ds = \omega e^{-\omega\rho x}\int_0^x f''(s)\,e^{\omega\rho s}\cdot\omega ds.$$

We begin now to make use of the properties of $f(s)$ in the complex s-plane. If a new variable is introduced by the transformation $s' = \omega s$, changing the integral along the axis of reals into an integral along a ray inclined at 120°, *$f''(s)$ is invariant under the transformation, being a single-valued function of s^3.* Dropping the accent after performing the transformation, we may write

$$\omega e^{-\omega\rho x}\int_0^x f''(s)\,e^{\omega\rho s}\cdot\omega ds = \omega e^{-\omega\rho x}\int_0^{\omega x} f''(s)\,e^{\rho s}\,ds.$$

Similarly,

$$\int_0^x f''(s)\cdot\omega e^{\omega^2\rho(s-x)}\,ds = \omega^2 e^{-\omega^2\rho x}\int_0^{\omega^2 x} f''(s)\,e^{\rho s}\,ds;$$

while, without any change of variable,

$$\int_0^x f''(s)\, e^{\rho(s-x)}\, ds = e^{-\rho x} \int_0^x f''(s)\, e^{\rho s}\, ds .$$

All three integrals are now expressed with a common integrand.

A further transformation is rendered possible by the fact that $f(s)$, *and so* $f''(s)$, *is an analytic function of* s. For any particular values of $x\,(0 \leqq x \leqq x_0)$ and ρ, let y be the point defined by the conditions

$$|y| = |x|, \qquad \arg y = \pi - \arg \rho,$$

so that

$$\rho y = -\,|\rho x| .$$

The straight path from 0 to ωx, for example, may be replaced by a broken line, extending from 0 to y and then from y to ωx. The two paths and the triangle bounded by them will be wholly interior to the region where the analytic character of $f(s)$ is assumed, so that the change of path will not change the value of the integral. That is,

$$\int_0^x f''(s) \cdot \omega^2 e^{\omega\rho(s-x)}\, ds = \omega e^{-\omega\rho x} \int_0^y f''(s)\, e^{\rho s}\, ds + \omega e^{-\omega\rho x} \int_y^{\omega x} f''(s)\, e^{\rho s}\, ds .$$

The paths from 0 to $\omega^2 x$ and from 0 to x may be modified in the same way, the first segment of the new path extending from 0 to y in each case. The result of all this is that

$$\begin{aligned} &\int_0^x f''(s)\, \delta_1(\rho\overline{x - s})\, ds \\ &\quad = \left[\delta(\rho x) \int_0^y + e^{-\rho x} \int_y^x + \omega e^{-\omega\rho x} \int_y^{\omega x} + \omega^2 e^{-\omega^2\rho x} \int_y^{\omega^2 x}\right] f''(s)\, e^{\rho s}\, ds . \end{aligned} \tag{15}$$

The notation is abbreviated, because the integrand on the right is the same throughout. It is understood that each path of integration is rectilinear.

Turn now to the other of the two integrals in A. In the first place,

$$\int_0^\pi f''(s)\, \delta_1(\rho\overline{\pi - s})\, ds = \left[\int_0^x + \int_x^\pi\right] f''(s)\, \delta_1(\rho\overline{\pi - s})\, ds . \tag{16}$$

The integral from x to π may be allowed to stand. The integral from 0 to x is similar to the one just treated. By the same line of reasoning it is found that

$$\begin{aligned} &\int_0^x f''(s)\, \delta_1(\rho\overline{\pi - s})\, ds \\ &\quad = \left[\delta(\rho\pi) \int_0^y + e^{-\rho\pi} \int_y^x + \omega e^{-\omega\rho\pi} \int_y^{\omega x} + \omega^2 e^{-\omega^2\rho\pi} \int_y^{\omega^2 x}\right] f''(s)\, e^{\rho s}\, ds . \end{aligned} \tag{17}$$

The analytic character of $f(s)$ is used again, but only in the same region as before.

If the value of A is calculated now, it is seen that the expression

$$\delta(\rho x)\,\delta(\rho\pi)\int_0^y f''(s)\,e^{\rho s}\,ds$$

occurs twice, with opposite signs, and is eliminated. *This is the critical point in the convergence proof.* The remaining terms are harmless by reason of their order of magnitude.

Let B stand for the right-hand member of (15), with the omission of the term containing $\delta(\rho x)$, C for the integral from x to π in (16), and D for the right-hand member of (17), without the term containing $\delta(\rho\pi)$. Then

$$\frac{A}{\rho^2\,\delta(\rho\pi)} = \frac{1}{\rho^2}\left[B - C\frac{\delta(\rho x)}{\delta(\rho\pi)} - D\frac{\delta(\rho x)}{\delta(\rho\pi)}\right].$$

The dependence of B and D on ρ and x is somewhat complicated because of the variability of y, but they are at any rate continuous functions of their arguments, so that there is no difficulty as to the existence of the integrals with regard to ρ.

In the first of the three integrals in B, as s goes along a straight path from y to x, ρs goes along a straight path from ρy to ρx, that is, from $-|\rho x|$ to ρx, for each value of ρ. The real part of ρx is certainly not less than $-|\rho x|$, and so $|e^{\rho s}|$ reaches its greatest value, as far as this integral is concerned, at the upper limit of integration. That is, $|e^{\rho s}| \leqq |e^{\rho x}|$, all along the path. Since the length of the path can not exceed $2x_0$, it follows that if M is the maximum of $|f''(s)|$ for $|s| \leqq x_0$,

$$\left|e^{-\rho x}\int_y^x f''(s)\,e^{\rho s}\,ds\right| \leqq 2Mx_0.$$

Similar reasoning applies to each of the other integrals in B; the definition of y was expressly chosen so as to make the factor $e^{\rho s}$ reach its maximum absolute value at the upper limit of integration in each case. By combination of the inequalities for the three integrals,

$$|B| \leqq 6Mx_0,$$

for all values of x in $(0, x_0)$ and for all values of ρ. Since B is uniformly bounded, and since the length of the path of the ρ-integration is proportional only to the first power of ρ, the integral of B/ρ^2 converges uniformly to zero.

In connection with C, it is enough to consider the ρ-integral along γ_n'', since $C\delta(\rho x)/[\rho^2\,\delta(\rho\pi)]$ takes on conjugate imaginary values for conjugate imaginary values of ρ. The absolute value of $\delta_1(\rho\pi - s)$ can not exceed $3|e^{\omega\rho(s-\pi)}|$, and as s is real here and always $\geqq x$, the expression just written down can not exceed $3|e^{\omega\rho(x-\pi)}|$. Then

$$|C| \leqq 3M\pi|e^{\omega\rho(x-\pi)}|, \tag{18}$$

the length of the path being $\leqq \pi$. On the other hand,

$$|\delta(\rho x)| \leqq 3|e^{-\omega\rho x}|,$$

whence

$$|C\delta(\rho x)| \leqq 9M\pi|e^{-\omega\rho\pi}|.$$

But

$$|\delta(\rho\pi)| > k|e^{-\omega\rho\pi}|,$$

where k has the same meaning as in the preceding section. So $C\delta(\rho x)/\delta(\rho\pi)$ is uniformly bounded, and when divided by ρ^2 becomes so small that its ρ-integral approaches zero.

In D, finally, the choice of y is effective in the same way as in B. The three integrals are less than or equal to

$$2Mx_0|e^{\rho x}|, \qquad 2Mx_0|e^{\omega\rho x}|, \qquad 2Mx_0|e^{\omega^2\rho x}|,$$

and the three terms containing them are less than or equal to

$$2Mx_0|e^{\rho(x-\pi)}|, \qquad 2Mx_0|e^{\omega\rho(x-\pi)}|, \qquad 2Mx_0|e^{\omega^2\rho(x-\pi)}|,$$

respectively, in absolute value. Again we may suppose that ρ is on γ_n''; then the second of the last three absolute values is at least as great as either of the others, and

$$|D| \leqq 6Mx_0|e^{\omega\rho(x-\pi)}|.$$

This inequality may be used in exactly the same way as the inequality (18) for $|C|$. It leads to the conclusion that the integral of $D\delta(\rho x)/[\rho^2\delta(\rho\pi)]$ uniformly approaches zero.

Thus the proof of Theorem III is completed.*

* In connection with the present account, attention should be called to the following facts, not brought out by Mr. Hopkins's own treatment: If $f(x)$ is analytic throughout the circle $|x| \leqq \pi$ in the complex plane ($x_0 = \pi$), the convergence of the integral of $A/[\rho^2\delta(\rho\pi)]$ to zero is uniform throughout the entire interval $(0, \pi)$, and nothing stands in the way of the uniform convergence of the series all the way out to the point π, except the term $3f(\pi)\delta(\rho x)/[\rho\delta(\rho\pi)]$. If $f(\pi) = 0$, this term vanishes, and the series is seen to be uniformly convergent for $0 \leqq x \leqq \pi$. It has been observed already that the sum of the series is zero in any case for $x = \pi$, so that this is the only case in which uniform convergence out to the end of the interval is possible.

GROUPS POSSESSING A SMALL NUMBER OF SETS OF CONJUGATE OPERATORS*

BY

G. A. MILLER

1. INTRODUCTION

Some properties of a group G can perhaps be most clearly presented by representing G as a kind of independent government. From this point of view every abstract group presents a distinct type of government so that there are as many forms of group governments as there are different abstract groups. The notion of regarding a group as a kind of government is not entirely new. Sophus Lie once divided all continuous transformation groups into two classes which he called *democratic* and *aristocratic* groups respectively,† and Felix Klein called conjugate operators of a group *equal rights* operators.‡

A common property of all finite groups is that every such group has one and only one identity. As a group operator this identity is a kind of figurehead having no effect on the operators with which it is combined. The uniqueness of the identity and the fact that as an operator it has no effect on the operators of G may suggest the appropriateness of regarding it as the king of G, and hence every finite group represents a type of a monarchy. Although the identity is a kind of figurehead it is the most prominent member of G and it is sometimes spoken of as the principal element of G. All the other elements of G are potentially equivalent to this king since some power of every element of a finite group is equal to the identity, but the other elements of G have normally properties which differ widely from those possessed by the identity.

The main objects of the present article are to establish a few general theorems relating to the possible number of sets of conjugate operators in a group of a given order, or the number of different classes of equal rights members of a group government, and to determine all the possible groups having a small number of complete sets of conjugate operators. In particular we shall determine an upper limit for the order of a discontinuous group having a given number of sets of conjugate operators so that every discontinuous

* Presented to the Society, December 27, 1918.

† S. Lie, Berichte der Gesellschaft der Wissenschaften zu Leipzig, vol. 47 (1895), p. 271.

‡ F. Klein, Mathematische Annalen, vol. 14 (1879), p. 430.

group which has a larger order must involve also a larger number of sets of conjugate operators. A striking difference between continuous and discontinuous is thus made clear since the former have always a non-denumerable set* of infinitesimal operators but these operators may appear in a small number of sets of conjugate. In particular, in the democratic groups noted above they appear in a single set.

2. General theorems

Let g represent the order of the group G. The number of operators in any one of the various complete sets of conjugate operators of G is equal to g divided by the order of the subgroup composed of all the operators of G which transform an arbitrary operator of the set into itself. In particular, the number of operators in any one of the complete sets of conjugate operators of G is a divisor of g, and a necessary and sufficient condition that a complete set of conjugate operators consists of a single operator is that this operator is invariant under G. When G is abelian the number of its complete sets of conjugate operators is equal to g and vice versa.

If $g_1, g_2, \cdots, g_k$ are the orders of the subgroups of G which have the property that each subgroup is composed of all the operators of G which transform into itself an operator of a complete set of conjugates and that there is one and only one such subgroup for each complete set of conjugate operators in G, then it results directly that

$$\frac{1}{g_1}+\frac{1}{g_2}+\cdots+\frac{1}{g_k}=1.$$

At least one of these denominators is equal to g since the identity constitutes a complete set of conjugate operators. When g is large the corresponding fraction must be small, and hence k must exceed a certain number. For instance, when $g = 100$ the value of k must exceed 4 since three unit fractions whose sum is as large as 99/100 can not have 100 for their least common denominator.

A maximum value of $g = g_k$ for a given value of k can readily be found as follows: If we assign a number to one of the symbols $g_1, g_2, \cdots, g_{k-1}$ the remaining symbols can assume their largest possible values when the value assigned to the first symbol is as small as possible, since the sum of the reciprocals of these numbers is fixed. Hence the largest possible value of g cannot exceed the number obtained for g_k by assigning to $g_1, g_2, \cdots, g_{k-1}$ in order, the values of the corresponding terms of the following series

$$2, 3, 7, 43, 1807, \cdots,$$

* *Encyclopédie des sciences mathématiques*, tome II, vol. 4, p. 176.

where each term after the first is obtained by multiplying together all the preceding terms and increasing by unity the product thus obtained, since $(n-1)/n+1/(n+1)$ is of the form $(m-1)/m$. This series increases very rapidly as the number of its terms increases, but it is finite whenever this number is finite. In particular, we have proved the following

THEOREM. *If the number k of complete sets of conjugate operators contained in a group is given then it is always possible to determine another number such that none of the possible discontinuous groups whose operators occur in exactly k complete sets of conjugates can have an order which exceeds this number.* As a special case of this theorem it may be noted that a necessary and sufficient condition that a discontinuous group is of finite order is that the number of its complete sets of conjugate operators is finite.*

In the following section it will be noted that the upper limit for the value of g resulting from the above theorem is usually much too large. When $k=2$ or 3 the theorem gives the exact value of g, but when $k=4$ it fixes 42 for an upper limit of the value of g while the actually largest value of $g=12$, and when $k=5$ the largest actual value of g is 60 while the upper limit obtained from the theorem is 1806.

A necessary and sufficient condition that G is solvable is that the identity can be attained by forming successive commutator subgroups of G. Let

$$C_1, C_2, \cdots, C_l = 1$$

represent such a successive set of commutator subgroups of a solvable group G and consider the corresponding quotient groups

$$G/C_1, G/C_2, \cdots, G/C_l \equiv G.$$

The number of complete sets of conjugate operators in each of these quotient groups, after the first, must be at least one larger than in the preceding quotient group since each of the successive commutator subgroups includes all those which follow it. In particular, if a solvable group contains a non-abelian commutator subgroup it must involve at least two more complete sets of conjugate operators than its commutator quotient group does.

As C_{l-1} is abelian the number of operators in one of its sets of conjugates under G cannot exceed the order of G/C_{l-1}. If the number of these sets in addition to the identity is α, the order of C_{l-1} cannot exceed 1 plus α times the order of G/C_{l-1}. For solvable groups this furnishes another proof of the

*Cf. E. Landau, Mathematische Annalen, vol. 56 (1903), p. 674. It may be noted that the statement of this theorem as given on p. 199 of vol. 1, second edition, Pascal's *Repertorium der höheren Mathematik*, 1910, is meaningless since the number of the distinct abstract groups of finite order may be regarded as finite irrespective of the number of conjugate sets of operators contained in such groups. The theorem established by Landau does not give a definite upper limit for the order of a group having a fixed number of sets of conjugate operators while such a limit is here established.

fact that a discontinuous group must be of finite order whenever the number of its complete sets of conjugate operators is finite. Some of these results may be stated in the form of a theorem, as follows:

THEOREM. *If the solvable group G has k complete sets of conjugate operators and if the largest solvable group which has $k - \alpha$, $\alpha \geqq 1$, complete sets of conjugate operators is of order q then the order of G cannot exceed $q(\alpha q + 1)$.* In particular, when all the operators besides the identity of any of the successive commutator subgroups are conjugate under G then this commutator subgroup must be C_{l-1} and its order is a power of a prime number.

3. DETERMINATION OF ALL THE GROUPS WHICH INVOLVE NO MORE THAN FIVE SETS OF CONJUGATE OPERATORS

The only finite group which possesses only two sets of conjugate operators, one in addition to the identity, is the group of order 2. In fact, each operator in the second set of conjugates of such a group must be transformed into itself by its powers and hence the number of conjugates in this set cannot exceed $g/2$. As this number must also be $g - 1$ it results that $g = 2$.

If a non-abelian group involves only three complete sets of conjugate operators its order cannot be a power of a prime since the commutator quotient group of a prime power group is always non-cyclic* and hence its order cannot be less than 4. The order of this non-abelian group must therefore be of the form $p^\alpha q^\beta$, p and q being prime numbers. As there are only two sets of conjugate operators besides the identity one of these sets must be composed of operators of order p while the other is composed of operators of order q. The former set contains at most $1/p$ of the operators of G while the latter contains at most $1/q$ of them. Hence it follows that g could not exceed 6 since

$$\frac{1}{g} + \frac{1}{p} + \frac{1}{q} = 1$$

is satisfied only when $g = 6$. It therefore results that *the non-cyclic group of order 6 is the only non-abelian group which involves three and only three complete sets of conjugate operators.*

When the non-abelian group G involves four and only four complete sets of conjugate operators its order cannot be a power of a prime for the reasons stated in the preceding paragraph. This order could not be divisible by three distinct prime numbers since this would imply that the operators in each of the sets of conjugates would be prime and that all the operators of the same prime order would be conjugate. As at least one of these primes p would exceed 3 there would have to be operators in G whose orders would be

* G. A. Miller, Annals of Mathematics, vol. 3 (1902), p. 180; H. Hilton, *Finite Groups*, 1908, p. 146.

divisible by $p-1$, viz., the operators which would transform an operator of order p into a power whose index is a primitive root of p. As these operators would be of composite order we have proved that if the operators of the non-abelian group G appear in exactly four complete sets of conjugates then g is of the form $p^\alpha q^\beta$, p and q being prime numbers. In particular, G must be solvable.

Since G is solvable and involves four complete sets of conjugate operators its commutator quotient group is either of order 2 or of order 3. In the latter case, the commutator subgroup is evidently abelian and its order cannot exceed 4. Hence G must be the tetrahedral group in this case. When the order of the commutator quotient group of G is 2 the commutator subgroup of G must again be abelian; for, if it were non-abelian the quotient group of G with respect to the second commutator subgroup would be non-abelian and it would involve three sets of conjugate operators. Hence it would be the non-cyclic group of order 6, and the order of G could not exceed 42. This order could clearly not be 42 since the group of isomorphisms of the group of order 7 is cyclic. It could not be 24 or 12 as can be readily verified. When the commutator subgroup of G is abelian and the commutator quotient group is of order 2 G must be the dihedral group of order 10 so that *the tetrahedral group and the dihedral group of order* 10 *are the only two non-abelian groups involving exactly four complete sets of conjugate operators.*

When all the operators of the non-abelian group G occur in five complete sets of conjugates G is not necessarily solvable as results directly from the icosahedral group, which involves exactly five complete sets of conjugate operators. We proceed to prove that no other simple group of composite order has this property, and hence we shall assume for the present that G is a simple group. The order of G cannot be divisible by four distinct prime numbers for the reasons given above in connection with the case when all the operators of G appear in four complete sets of conjugates. Hence we may assume that $g = p^\alpha q^\beta r^\gamma$, where p, q, and r represent distinct prime numbers.

It is easy to prove that these prime numbers must be 2, 3, and 5. In fact, one of them must be 2 since at least one of the sets of conjugates is necessarily composed of all the operators of G having the same odd prime order. Moreover, g must be divisible by 4 since every group whose order is the double of an odd number is composite. The number of the operators of order 2 in G can therefore not exceed $g/4$. If both of the other prime factors were greater than 3 G would involve operators of more than four different orders besides the identity. The order of G must therefore be of the form $2^\alpha 3^\beta p^\gamma$, p being a prime number greater than 3.

If all the operators of order $p > 5$ were conjugate the number of operators of each of the orders 2 and $p-1$ contained in G could not exceed $g/(p-1)$

while the operators of orders 3 and p could not exceed $g/3$ and g/p respectively. Since

$$\frac{2}{p-1}+\frac{1}{3}+\frac{1}{p}+\frac{1}{g}<1$$

for $g \geqq 60$ this is impossible. If all the operators of order p were conjugate when $p = 5$ the operators of order 4 contained in G would be transformed into their inverses by operators of order 2 and therefore each of the latter operators would be transformed into itself by at least 8 operators of G. Hence the number of operators of orders $2, 4, 3,$ and 5 contained in G could not exceed

$$g/8,\ g/4,\ g/3, \text{ and } g/5$$

respectively. As

$$\frac{1}{8}+\frac{1}{4}+\frac{1}{3}+\frac{1}{5}+\frac{1}{g}<1$$

when $g \geqq 60$, the group is impossible.

All the operators of order p contained in G are therefore found in two complete sets of conjugates, and the following equation

$$\frac{1}{4}+\frac{1}{3}+\frac{2}{p}+\frac{1}{g}=1$$

must be satisfied for some value of $g \geqq 60$. This is clearly only possible when $p = 5$ and $g = 60$. That is, *there is one and only one simple group of composite order which has the property that all of its operators appear in five complete sets of conjugates; viz., the icosahedral group.* From this theorem it results directly that no composite insolvable group involves exactly five complete sets of conjugate operators.

In what follows it will be assumed that G is solvable and that it involves five and only five complete sets of conjugate operators. The order of the commutator quotient group of G is therefore one of the three numbers $2, 3, 4$. When this order is 4, the commutator subgroup of G must be abelian and have an order which cannot exceed $4 + 1 = 5$. When this order is 5 G is the metacyclic group of order 20. When this order is 2 G is either the octic group or the quaternion group. There are therefore *three and only three non-abelian groups whose operators appear in exactly five complete sets of conjugates and whose commutator quotient group is of order* 4, *viz., the octic group, the quaternion group, and the metacyclic group of order* 20.

When the commutator quotient group of G is of order 3 its commutator subgroup must again be abelian, for if this commutator subgroup were non-abelian the quotient group of G with respect to its second commutator subgroup would be non-abelian and would involve four complete sets of conjugate operators. As its commutator quotient group would be of order 3 it would

be the tetrahedral group. Moreover, the second commutator subgroup of G would be abelian since all of its operators besides the identity would be conjugate under G. The order of this subgroup could therefore not exceed 13. It could not be 13 since the group of isomorphisms of the group of order 13 is cyclic. For a similar reason it could not be 7 or 5. If it were 4, G would be of order 48 and its subgroup of order 16 would be non-abelian, contain 12 operators of order 4 which would be conjugate under G, and have three invariant commutators of order 2. As this is impossible the second commutator subgroup of G could not be of order 4. As this order could clearly not be 2 or 3 it results that the commutator subgroup of G must be abelian and that *the non-cyclic group of order* 21 *is the only group whose operators appear in exactly five complete sets of conjugates and whose commutator quotient group is of order* 3.

It remains to consider the case when the solvable non-abelian group G involves exactly five complete sets of conjugate operators and when the commutator quotient group of G is of order 2. The special case when the commutator subgroup of G is abelian is easily disposed of since G could then not exceed 14 and the dihedral group of order 14 is evidently the only one that satisfies these conditions. Hence we shall assume in what follows that the commutator subgroup H of G is non-abelian. We proceed to prove that the commutator subgroup of H, or the second commutator subgroup of G, must be abelian.

If the commutator subgroup of H were non-abelian the quotient group of G with respect to the commutator subgroup of this commutator subgroup would involve four complete sets of conjugates and have an $(\alpha, 1)$ isomorphism with the non-cyclic group of order 6. As neither of the two non-abelian groups involving exactly four complete sets of conjugate operators has the latter property the commutator subgroup of H must be abelian. Moreover, the quotient group of G with respect to this abelian subgroup must be non-abelian and cannot involve more than four complete sets of conjugate operators. As this quotient group involves a subgroup of index 2 it is either the non-cyclic group of order 6 or the non-cyclic group of order 10.

It could not be the non-abelian group of order 10 since all the operators of G corresponding to the same non-identity operator of this quotient group would be conjugate under G and the subgroup of G corresponding to the identity in this quotient group would have an order which could not exceed 11. Hence it remains only to consider the case when the quotient group of G with respect to its second commutator subgroup is the non-abelian group of order 6. Since this commutator subgroup is abelian and its order could clearly not be 7 its order must be 4 and G must be the group of the cube. *There are therefore exactly seven non-abelian groups whose operators occur in five and only five complete sets of conjugates.* One of these is insolvable and

the remaining six are solvable. Three of the latter have a commutator quotient group of order 4, in two of them this quotient group is of order 2, and in the remaining one it is of order 3.

4. Complete sets of conjugate operators under the group of isomorphisms

A group G may contain operators which are not conjugate under G but are conjugate under the group of isomorphisms of G. These operators may be said to be implicitly equal rights operators while those which are conjugate under G are explicitly equal rights operators. In the present section we shall call operators of G conjugate if they are conjugate under the group of isomorphisms of G, and shall especially consider those groups which do not involve more than three complete sets of such conjugate operators. The identity clearly constitutes a complete set under the group of isomorphisms of G as well as under G itself.

A necessary and sufficient condition that G contains only one complete set of conjugate operators besides the identity is that it is an abelian group of prime power order and of type $(1, 1, 1, \cdots)$, and a necessary and sufficient condition that an abelian group has exactly two complete sets of conjugate operators besides the identity is that it is a prime power group and of type $(2, 2, 2, \cdots)$. Hence we shall assume in what follows that G is non-abelian and that it has exactly two complete sets of conjugate operators besides the identity.

The order g of G can clearly not be divisible by more than two distinct prime numbers and hence G must be solvable. If g is of the form $p^\alpha q^\beta$, p and q being distinct primes and $\alpha, \beta \geqq 1$, the commutator subgroup C of G must be abelian and of type $(1, 1, 1, \cdots)$ and must be a Sylow subgroup of G. Hence we may assume that the order of C is p^α and that the order of the commutator quotient group is q^β. The latter group is evidently also abelian and of type $(1, 1, 1, \cdots)$. It results therefore that G contains only one Sylow subgroup of order p^α and that its Sylow subgroups of order q^γ are abelian and of type $(1, 1, 1, \cdots)$.

As none of the operators of C, except the identity, can be transformed into itself by an operator of order q contained in G it results that $p^\alpha - 1$ is divisible by q^β. A subgroup of order q^β cannot be transformed into itself by any operator besides the identity of the subgroup of order p^α. In fact, if a subgroup of order q^β were transformed into itself by a group of order p^γ, $\gamma > 0$, G would contain a subgroup of order $p^\gamma q^\beta$ which would involve only $p^\gamma - 1$ operators whose orders are divisible by p. Hence this subgroup would involve an invariant subgroup of each of the orders p^α and q^β. As these subgroups have only the identity in common every operator of the one

would be commutative with every operator of the other. This is impossible since G involves no operator whose order is pq. Hence G contains p^α subgroups of order q^β.

That the value of β must be 1 may be proved as follows: The group of isomorphisms of the abelian group of order p^α and of type $(1, 1, 1, \cdots)$ contains at least one cyclic group of order $p^\alpha - 1$. In fact, when $\alpha > 1$ it is well known that this group of isomorphisms contains more than one such cyclic group. As the order of this group of isomorphisms is

$$(p^\alpha - 1)(p^\alpha - p)(p^\alpha - p^2)\cdots(p^\alpha - p^{\alpha-1})$$

it is clear that its subgroup of order q^β is cyclic unless $p^\gamma - 1$, $\gamma < \alpha$, is divisible by q. When this subgroup is non-cyclic it must involve operators which are commutative with some of the operators of order p in the group of order p^α. Since the group of order q^β in the present case is known to be non-cyclic whenever $\beta > 1$ we have established the following

THEOREM. *If all the operators besides the identity of a non-abelian group belong to two sets of conjugates under its group of isomorphisms the order of this group is either of the form p^α or of the form $p^\alpha q$, p and q being prime numbers.*

As an illustration of an infinite system composed entirely of groups having separately only two systems of conjugate operators besides the identity under their respective groups of isomorphisms we may cite the groups of order $2p^\alpha$, $p > 2$, obtained by extending every abelian group of order p^α and of type $(1, 1, 1, \cdots)$ by means of an operator of order 2 which transforms each of the operators of this abelian group into its inverse. On the other hand, it is easy to construct groups of order $p^\alpha q$ involving only operators of order p and q besides the identity but having more than three systems of conjugate operators under their groups of isomorphisms.

To obtain such a group we may assume $p = 2$ and q any prime number of the form $2^\alpha - 1$, $\alpha > 2$. It is known that the group of isomorphisms of the abelian group of order 2^α and of type $(1, 1, 1, \cdots)$ is simple and hence it cannot involve negative substitutions. This group of order $2^\alpha q$ can be represented as a substitution group of degree $2^\alpha - 1$. As it contains no negative substitutions it cannot contain any substitutions which transform the cycles of order q into their inverses. Hence the groups of order $2^\alpha q$, where $q = 2^\alpha - 1$ and $\alpha > 2$, which involve only operators of orders 2 and q in addition to the identity, do not permit isomorphisms in which each operator of order q is made to correspond with every other operator of this order. The smallest group coming under this theorem is of order 56.

When a non-abelian group G of order p^α is such that all of its operators appear in three sets of conjugates under its group of isomorphisms all of its commutators besides the identity must constitute one such set. Since at least

one non-identity commutator of every non-abelian prime power group is invariant under the group all the commutators of G must be invariant operators under G. Hence the commutator subgroup of G must coincide with its central and all the operators of G besides the identity must be of order p or of order p^2. When $p = 2$ operators of the latter order must occur in G but this is not necessarily true when $p > 2$.

We shall first consider the case when operators of order p^2 occur in G. If p^β and p^γ represent respectively the orders of the commutator subgroup and of the commutator quotient group of G it is easy to prove that $\beta = \gamma$ when $p > 2$, and that β is a divisor of γ when $p = 2$. In fact, when $p > 2$ the pth power of the product of any two operators of order p^2 contained in G is the product of the pth powers of these operators since their commutator is invariant and of order p.* Hence all the operators of G which have the same pth power must appear in the same coset with respect to the central of G. Moreover, all the operators of order p contained in the central of G must be pth powers of operators of order p^2 contained in G since the operators of order p are conjugate under the group of isomorphisms of G. These conditions imply the following

THEOREM. *If a non-abelian group of order p^α, $p > 2$ involves operators of order p^2 and has the property that all of its operators occur in three complete sets of conjugates under its group of isomorphisms then its commutator subgroup is of order $p^{\alpha/2}$.*

When $p = 2$ it is not necessarily true that $\beta = \gamma$ as results directly from the quaternion group. The squares of the operators in the same coset with respect to the commutator subgroup are evidently equal to each other and the number of such different cosets whose operators have the same square is clearly the same for the various operators of order 2 contained in G since all of the latter are conjugate under the group of isomorphisms of G. This implies that

$$\frac{2^\gamma - 1}{2^\beta - 1}$$

is an integer, which is possible only when γ is a multiple of β.

It is not difficult to prove that α is necessarily odd whenever G involves operators of order p^2 and $p > 2$. In fact, the independent generators of the commutator subgroup of G are the commutators of pairs of operators corresponding to independent generators of the quotient group of G. Hence the number of the pairs of the latter generators must be divisible by the number of the former generators since the latter number is equal to the number of the independent generators in the quotient group of G. As the first of these numbers is divisible by the second only when β is odd there results the

* G. A. Miller, Annals of Mathematics, vol. 3 (1902), p. 180.

THEOREM. *If a non-abelian group of order p^α, $p > 2$, involves operators of order p^2 and has the property that all of its operators appear in three sets of conjugates under its group of isomorphisms then α is the double of an odd number.*

The preceding considerations establish the fact that the number of the independent generators of the commutator subgroup of G is always a divisor of the number of the pairs of independent generators in the commutator quotient group of G, but when G does not involve operators of order p^2 or when $p = 2$ the order of the commutator subgroup of G need not be equal to the order of its commutator quotient group as results directly from the fact that there is one and only one group of order p^3, p any prime number, which has the property that all of its operators appear in three sets of conjugates under its group of isomorphisms. When $p = 2$ this is the quaternion group and when $p > 2$ it is the non-abelian group which involves no operator of order p^2. There is also one and only one group of order p^4 for every value of p which has the given properties but this group is necessarily abelian.

When the commutator subgroup of G is of order $p > 2$ the independent generators of the commutator quotient group of G can be so selected that the operators of G which correspond to any one of them are commutative with those corresponding to each of the others save one. Hence the commutator quotient group of G must be of even order while α is an odd number. Moreover α can evidently be an arbitrary odd number > 1 for a G having a commutator subgroup of order p.* All of these groups involve only operators of order p besides the identity since all the non-invariant operators of G are conjugate under its group of isomorphisms. It results therefore that *there is at least one group of order p^α, $\alpha > 1$ and $p > 2$, which has the property that all of its operators appear in three sets of conjugates under its group of isomorphisms.*

When $p = 2$ and 2^β is the order of the commutator subgroup of G while 2^γ is the order of its commutator quotient group it was noted above that γ is divisible by β and that β must also be a divisor of the number of pairs that can be formed with γ objects whenever G is non-abelian. As $\alpha = \beta + \gamma$ it results directly that α cannot be equal to 5 since γ cannot excel 2 when $\beta = 1$.† That is, the smallest order of the form 2^α, $\alpha > 1$, for which there is no group whose operators appear in three sets of conjugates under the group of isomorphisms is 32.

* J. W. A. Young, American Journal of Mathematics, vol. 15 (1893), p. 171.

† Miller, Blichfeldt, Dickson, *Finite Groups*, 1916, p. 127.